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FACIAL STRUCTURES FROM THE ORDER - THEORETICAL POINT OF VIEW

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The aim of this paper is to outline a large generalization of the convexity theory, based on the study of certain order relations, geometrically determined. That will allow us to bring together appearently unrelated facts and results and to explain the similarity between several domains of functional analysis.

The present paper has been circulated in the early 90's as a preprint, entitled *Lectures of Alfsen - Effros theory*.

1. ALFSEN-EFFROS TYPE ORDER RELATIONS

Let E be a Banach space over the field $\mathbb{K}$ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$).

1.1 DEFINITION (see [N2]). An order relation « on E is said to be of **Alfsen-Effros type (abbreviated, « is an AE-order relation)** provided that the following conditions are satisfied:

AE1) $x \ll y$ implies $y-x \ll y$;

AE2) $x \ll y$ implies $\alpha x \ll \alpha y$ for every $\alpha \in \mathbb{K}$;

AE3) $0 \leq \alpha \leq \beta$ in $\mathbb{R}$ implies $\alpha x \ll \beta x$ for every $x \in E$;

AE4) If $x_1 \ll y_1$, $x_2 \ll y_2$ and $y_1 \ll y_1 + y_2$ then $x_1 \ll x_1 + x_2$ and $x_1 + x_2 \ll y_1 + y_2$;

AE5) $x + y \ll 2y$ implies $\|x\| \leq \|y\|$;

AE6) $x_a \ll y$ $(a \in A)$ and $\|x_a - x\| \to 0$ implies $x \ll y$.

Clearly, the definition above can be adapted in an evident manner for locally convex spaces with a specified system of seminorms. Also, one can rephrase the conditions AE1) - AE6) above in terms of codirection by letting

$x \parallel y$ (i.e., x and y are codirectional) if and only if $x \ll x+y$.

The next proposition collects immediate consequence of Definition 1.1.

1.2 PROPOSITION. *Let E be a Banach space endowed with an AE-order relation «. Then:*

i) $0 \ll x$ *for every* $x \in E$;

ii) $x \ll y$ *and* $-x \ll y$ *implies* $x = 0$;

iii) $x \ll y$ *implies* $\|x\| \ll \|y\|$;

iv) $x \parallel y$ *implies* $y \parallel x$ *and* $\alpha x \parallel \beta y$ for every $\alpha, \beta \geq 0$.

From Proposition 1.2 i) we infer that an AE-order relation is not compatible with the linear structure. As we shall show in the next sections the AE-order relations are very suitable to describe the geometry of the unit ball of the underlying Banach space.

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1.3 PROPOSITION. *The one-dimensional Banach space* $\mathbb{K}$ *admits only one AE-order relation, namely*

$$x \ll y \text{ if and only if } x = \alpha y \text{ for a suitable } \alpha \in [0,1].$$

Proof. We shall consider here only the complex case. Suppose that $r_1 e^{i\theta_1} \ll r_2 e^{i\theta_2}$, where $r_1, r_2 > 0$ and $\theta_1, \theta_2 \in [0, 2\pi)$. By proposition 1.2 iii), we obtain $r_1 \leq r_2$ and condition AE2) above leads us to the case where $e^{i\theta} \ll r$ for some $r \geq 1$ and $\theta \in [0, 2\pi)$. We have to prove that $\theta = 0$.

For, $e^{i\theta} \ll r$ yields $e^{in\theta} \ll r^n$ for every $n \in \mathbb{N}$, so by AE1) and Proposition 1.1.2 iii) above we obtain $|r^n - e^{in\theta}| \leq r^n$ (and thus $1 - 2r^n \cos n\theta \leq 0$)for every $n \in \mathbb{N}$, a contradiction. ■

1.4 COROLLARY. *If* E *is a Banach space endowed with an AE-order relation* $\ll$ *then*

$$\alpha \ll \beta \text{ in } \mathbb{K} \text{ and } x \ll y \text{ in } E \text{ imply } \alpha x \ll \beta y.$$

Deeper examples come in connection with the facial structure of a Banach space E and they were first considered by Alfsen and Effros [AE]:

$$x \ll_L y \text{ if and only if } \|y\| = \|x\| + \|y - x\|.$$

(If E is strictly convex this means that the points $0, x, y$ are colinear and x is between 0 and y);

$$x \ll_M y \text{ if and only if every closed ball containing } 0 \text{ and } y \text{ contains also } x.$$

Notice that

$$x \ll_M y \text{ if and only if } \|x-z\| \leq \|z\| \vee \|y-z\| \text{ for every } z.$$

The verification of AE1)-AE6) above for $\ll_L$ needs only the triangle inequality; e.g., AE4) can be deduced as follows. By hypotheses,

$$\|y_1\| = \|x_1\| + \|y_1 - x_1\|, \ \|y_2\| = \|x_2\| + \|y_2 - x_2\|$$

and $\|y_1 + y_2\| = \|y_1\| + \|y_2\|$. Then

$$\|y_1 + y_2\| \leq \|y_1 + y_2 - x_1 - x_2\| + \|x_1 + x_2\| \leq$$

$$\leq \|y_1 - x_1\| + \|x_1\| + \|y_2 - x_2\| + \|x_2\| \leq$$

$$\leq \|y_1\| + \|y_2\| = \|y_1 + y_2\|$$

which implies that $x_1 + x_2 \ll_L y_1 + y_2$ and $x_1 \ll_L x_1 + x_2$.

The geometric meaning of $\ll_M$ is much more involving. For example, the condition AE1) means that symmetrical balls contain symmetrical points.

Except for AE4), the fact that $\ll_M$ is indeed an *AE*-order relation is simply routine. As concerns AE4), we know a simple argument only for the following statement $x_1 \ll_M y_1$, $x_2 \ll_M y_2$ and $y_1 \ll_M y_1 + y_2$ implies $x_1 + x_2 \ll_M y_1 + y_2$. In fact, our hypotheses are

$$\|x_1 - z\| \leq \|z\| \vee \|y_1 - z\|$$

$$\|x_2 - z\| \leq \|z\| \vee \|y_2 - z\|$$

$$\|y_2 - z\| \leq \|z\| \vee \|y_1 + y_2 - z\|$$

for every $z \in E$. Then by applying succesively these relations, we infer that

$$\|x_1 + x_2 - z\| \leq \|z - x_2\| \vee \|x_2 + y_1 - z\| \leq$$

$$\leq \|z\| \vee \|y_2 - z\| \vee \|y_1 - z\| \vee \|y_1 + y_2 - z\| \leq$$

$$\leq \|z\| \vee \|y_1 + y_2 - z\|$$

i.e., $x_1 + x_2 \ll_M y_1 + y_2$.

The other part of AE4) combines Theorem 1.9 below with the remark that only the real structure of E is involved in AE4). We shall need some background on Choquet's theory. The details omited are to be found in [Ph]. Another proof of the fact that $\ll_M$ is an AE-order relation appears in [NV2].

For K a compact convex subset of a locally convex Hausdorff space Z, we shall denote by $A(K, \mathbb{R})$ the Banach space (endowed with the sup norm) of all continuous real affines functions $f: K \to \mathbb{R}$.

1.5 LEMMA. *For each* $\varepsilon > 0$ *and each* $h \in A(K, \mathbb{R})$ *there exist* $z' \in Z'$ *and* $\alpha \in \mathbb{R}$ *with* $\| h - (z' + \alpha) \mid K \| < \varepsilon$.

Proof. Consider the following two subsets of $Z \times \mathbb{R}$:

$$M_1 = \{(x, r) \mid x \in K, h(x) = r\} \text{ and } M_2 = \{(x, r) \mid x \in K, h(x) = r+\varepsilon\}.$$

M_1 and M_2 are both compact, convex, non-empty and $M_1 \cap M_2 = \varnothing$. By Hahn-Banach separation theorem, there exist a continuous linear functional L on $Z \times \mathbb{R}$ and a real number λ such that $\sup L(M_1) < \lambda < \inf L(M_2)$. Then we can define a function g on Z by the formula $L(x,g(x)) = \lambda$ i.e., $L(x,0) + g(x) \cdot L(0,1) = \lambda$. Then $g \in Z'|K + \mathbb{R}$ and $h \leq g \leq h + \varepsilon$. ■

1.6 COROLLARY. *Let E be a Banach space and K the unit ball of E' endowed with the w'-topology. Then*

$$A(K,\mathbb{R}) = \{x \mid K + r \mid x \in E, r \in \mathbb{R}\}.$$

Let $\mathscr{P}(K)$ be the set of all probability measures on K. We shall say that a measure $\mu \in \mathscr{P}(K)$ *represents* the point x of K provided that

$$\mu(h) = h(x) \text{ for every } h \in A(K, \mathbb{R}).$$

1.7 LEMMA. *A point x of K is extremal for K if and only if the only measure $\mu \in \mathscr{P}(K)$ which represents x is the Dirac measure concentrated in x.*

Given a function $f \in C(K,\mathbb{R})$, we shall denote by $\hat{f}$ its *upper* enveloppe,

$$\hat{f}(x) = \inf \{h(x) \mid h \in A(K,\mathbb{R}), h \geq f\}, x \in K.$$

1.8 LEMMA. *For every* $f \in C(K,\mathbb{R})$,

$$\hat{f}(x) = \sup \{\mu(f) \mid \mu \textit{ represents } x\}.$$

1.9 THEOREM. *Let E be a real Banach space and let* $\mathrm{E}\times E'$ *be the set of all extreme points of the unit ball K of E'. Then the following assertions are equivalent for x and y two elements of E:*

i) $x \ll_M y$;

ii) $x \leq \widehat{y \vee 0}$ *on* K;

iii) *For every* $e' \in \mathrm{Ex}\, E'$ *either* $0 \leq e'(x) \leq e'(y)$ *or* $e'(y) \leq e'(x) \leq 0$.

Proof. i) $\Rightarrow$ ii). Let $\overline{B}_r(z)$ be the closed ball in E of center z and radius $r > 0$. The fact that $v \in \overline{B}_r(z)$ is equivalent to $v \leq z + r$ on K. Consequently 0 and y belong to $\overline{B}_r(z)$ if and only if $0 \vee y \leq z + r$ and thus i) is equivalent to the assertion that if $z \in E$ and $r > 0$ are such that $0 \vee y \leq z + r$ then $x \leq z + r$. By Corollary 1.6 above, every continuous affine majorant of $0 \vee h$ has the form $z + r$, so the latter assertion is equivalent to ii).

The implication ii) $\Rightarrow$ iii) follows from Lemmata 1.7 and 1.8 above.

iii) $\Rightarrow$ 1). Let $\sum \lambda_n e'_n$ a convex combination of elements e'_n of Ex E'. By hypotheses, for every n there exists an $\alpha_n \in [0,1]$ such that $e'_n(x) = \alpha_n e'_n(y)$. Consequently, if $0, y \in \overline{B}_r(z)$ then

$$\left|\sum \lambda_n e'_n(x-z)\right| \leq \sum \lambda_n \alpha_n \left|e'_n(y-z)\right| + \sum \lambda_n (1-\alpha_n)\left|e'_n(z)\right| \leq$$
$$\leq \sum \lambda_n \alpha_n r + \sum \lambda_n (1-\alpha_n) r = r$$

and thus by Krein-Milman Theorem it follows that $x \in B_r(z)$. ■

In 1983 the author has remarked (see [N2]) that the conditions AE1)-AE6) above are fulfilled in the context of Banach lattices by the order relation $\ll_o$,

$$x \ll_o y \text{ if and only if } |y| = |x| + |y - x|.$$

1.10 LEMMA. *Let E be a Banach lattice. Then the following assertions are equivalent*:

i) $x \ll_o y$;

ii) $x^+ \leq y^+$ *and* $x^- \leq y^-$;

iii) *Every order interval* $[u,v]$ *of E containing* 0 *and* y *contains also* x.

Proof. i) $\Rightarrow$ ii). In fact, from $y^\pm \leq x^\pm + (y - x)^\pm$ and $|y| = |x| + |y - x|$ it follows that $y^\pm = x^\pm + (y - x)^\pm$.

ii) $\Rightarrow$ i). Clearly, $y - x = (y^+ - x^+) - (y^- - x^-)$ and $0 \leq (y^+ - x^+) \wedge (y^- - x^-) \leq y^+ \wedge y^- = 0$. Consequently $(y-x)^\pm = y^\pm - x^\pm$.

ii) $\Rightarrow$ iii). If 0 and y are in $[u,v]$ then $0 \leq y^+ \leq v$ and $u \leq -y^- \leq 0$. Consequently, if $x^\pm \leq y^\pm$ then $0 \leq x^+ \leq v$ and $u \leq -x^- \leq 0$, which implies that $u \leq x \leq v$.

iii) $\Rightarrow$ ii). 0 and y belong to $[-y^-, y^+]$; if x belongs also to this interval then $x \leq y^+$ and $-y^- \leq x$. Consequently $x^\pm \leq y^\pm$. ■

We can extend the definition of $\ll_o$ to cover the class of all regularly ordered Banach spaces (in the sense of Davies [D]).

By an *ordered Banach space* we shall mean any real Banach space E endowed with a closed cone E_+ which is convex, proper and $E = E_+ - E_+$; on E we consider the ordering associated to E_+ i.e., $x \le y$ if and only if $y - x \in E_+$. An ordered Banach space E is said to be **regularly ordered** provided that the following two conditions are satisfied:

R1) If $-y \le x \le y$ then $\|x\| \le \|y\|$;

R2) If $x \in E$ and $\varepsilon > 0$ then there exists a $y_\varepsilon \in E$ such that $-y_\varepsilon \le x \le y_\varepsilon$ and $\|y_\varepsilon\| \le \|x\| + \varepsilon$.

Examples of regularly ordered Banach spaces are:

– the Banach lattices;

– the ordered Banach space A_{sa} of all self-adjoint elements of a C^* - algebra;

– the Banach space $A(K,\mathbb{R})$ when endowed with the pointwise order;

– any ordered Banach space E with a strong order unit $u > 0$ $\left(i.e., E = \bigcup_{n=0}^{\infty}[-nu, nu]\right)$, when the norm coincides with the norm associated to u,

$$\|x\|_u = \inf\{\lambda \mid \lambda \ge 0, -\lambda u \le x \le \lambda u\}.$$

On a regularly ordered Banach space E we consider the AE-order relation $\ll_o$ given by

$x \ll_o y$ if and only if every order interval $[u,v]$ containing 0 and y contains also x.

The proof that $\ll_o$ is indeed an AE-order relation needs the fact that every regularly ordered Banach space is locally a space $A(K,\mathbb{R})$. The basic ingredient is the following

1.11 THEOREM (see [Kad2]). *Let E be an ordered Banach space with a strong order unit $u > 0$ such that the norm on E is the norm associated to u. Then the set*

$$K = \{x' \mid x' \in E', x'(u) = 1 = \|x'\|\}$$

of all states on E is w'-compact, convex and the map $T : E\, A(K,\mathbb{R})$ given by

$$(Tx)x' = x'(x),\ x \in E,\ x' \in K$$

is an algebraic, isometric and order isomorphism.

Proof. A well known theorem due to Alaoglu shows that K is w'-compact. Clearly, it is also convex. If $0 \le x \le nu$ for some $n \in \mathbb{N}$, then $\|nu - x\| \le n$, so for each $x' \in K$ we have $|x'(nu)-x'(x)| \le n$. Consequently $x'(x) \ge 0$ for $x \ge 0$.

For $x \in E$, put $\alpha(x) = \inf\{\lambda \mid \lambda \ge 0, x \le \lambda u\}$. Then $\alpha(\lambda x) \ge \lambda\cdot\alpha(x)$ for every $\lambda \in \mathbb{R}$. Suppose that x and u are linearly independent. Then the functional x' given on Span$\{x,u\}$ by the formula $x'(\lambda x + \mu u) = \lambda\cdot\alpha(x) + \mu$ is linear and satisfies the relation $x'(u) = 1 = \|x'\|$; via Hahn-Banach extension theorem x' gives rise to an element of K. The above reasoning shows that T is a linear isometry and $x \ge 0$ if and only if $Tx \geq 0$. Then in order to prove that T is an onto map it suffices to shows that the image of T is dense in $A(K,\mathbb{R})$. In fact, by Lemma 1.5 above, for every $h \in A(K,\mathbb{R})$ and every $\varepsilon > 0$ there exist $x \in E$ and $r \in \mathbb{R}$ such that $\| h - T(x + ru) \| \leq \varepsilon$. ■

We can now prove that every regularly ordered Banach space E is locally a space $A(K,\mathbb{R})$. In fact, for every $x \in E$, $x > 0$, we can consider the **principal ideal generated by** x,

$$E_x = \{y \mid y \in E \ -\alpha x \leq y \leq \alpha x \text{ for a suitable } \alpha > 0\}$$

endowed with the induced order and the norm $\| \ \|_x$ associated to the strong order unit x,

$$\|y\|_x = \inf\{\alpha \mid -\alpha x \leq y \leq \alpha x\}.$$

By Theorem 1.11, E_x is algebraic, isometric and order isomorphic to a space $A(K_x,\mathbb{R})$ for a suitable compact Hausdorff convex space K_x. Since $\pm y \leq \|y\|_x \cdot x$ for every x with $x \geq \pm y$, the regularity of the norm of E yields

$$\|y\| = \inf\{\|y\|_x \cdot \|x\| \mid x \geq \pm y\}.$$

1.12 LEMMA. *The following assertions are equivalent for x and y two elements of a space $A(K,\mathbb{R})$:*

i) $x \ll_o y$;

ii) $x \ll_M y$;

iii) *For each $s \in K$, either* $0 \leq x(s) \leq y(s)$, *or* $y(s) \leq x(s) \leq 0$.

Proof. The equivalence ii) $\Leftrightarrow$ iii) follows from Theorem 1.9 above. Clearly, i) $\Rightarrow$ ii).

iii) $\Rightarrow$ i). Let $0, y \in [u,v]$ and $s \in K$. If $0 \leq x(s) \leq y(s)$ then $u(s) \leq 0 \leq x(s) \leq y(s) \leq$ $\leq v(s)$. If $y(s) \leq x(s) \leq 0$ then $u(s) \leq y(s) \leq x(s) \leq 0 \leq v(s)$. Consequently $x \in [u,v]$. ■

1.13 COROLLARY. *Let E be a regularly ordered Banach space and $x,y \in E$. Then the following assertions are equivalent:*

i) $x \ll_o y$;

ii) *If $z \in E_+$ and $y \in E_z$ then $x \in E_z$ and $x \ll_M y$ in E_z;*

iii) *There exists a $z_0 \in E_+$ such that for every $z \geq z_0$ with $y \in E_z$ we have $x \in E_z$ and $x \ll_M y$ in E_z.*

1.14 PROPOSITION. *The order relation $\ll_o$ is an AE-order relation.*

Proof. The fact that $\ll_o$ is indeed an order relation that satisfies the conditions AE1)-AE4) & AE6) follows Lemma 1.12 and Corollary 1.13. Now suppose that $x + y \ll_o 2y$ and let $z \geq \pm y$. By Corollary 1.13 it follows that $x + y \ll_M 2y$ in E_z, which implies that $\|x\|_z \leq \|y\|_z \leq 1$. Particularly, $z \geq \pm x$. Then

$$\|y\| = \inf_{z \geq \pm y} \|y\|_z \cdot \|z\| \geq \inf_{z \geq \pm y} \|x\|_z \cdot \|x\| \geq$$

$$\geq \inf_{z \geq \pm x} \|x\|_z \cdot \|x\| = \|x\|. \ ■$$

It is possible to introduce *regularly ordered complex Banach spaces* by complexification. See [LTz2] or [S2] for the particular case of complex Banach lattices. The order relation $\ll_o$ can be adapted easily to this context.

2. *AE*-ORDER RELATIONS ASSOCIATED TO VECTOR NORMS

Let E be a Banach space. By a **vector norm on** E we mean any map φ from E into a Banach lattice X such that:

N1) $\varphi(x)$ for every $x \in E$; $\varphi(x) = 0$ if and only if $x = 0$

N2) $\varphi(\alpha x) = \alpha\, \varphi(x)$ for every $\alpha \in \mathbb{K}$, $x \in E$

N3) $\varphi(x + y) \leq \varphi(x) + \varphi(y)$ for every $x,y \in E$.

We shall assume in addition that φ is also an isometry i.e.,

$$\|\varphi(x)\| = \|x\| \text{ for every } x \in E.$$

Then we can associate to φ the following two order relations on E:

$$x \ll_{L,\varphi} y \text{ if and only if } \varphi(y) = \varphi(x) + \varphi(y - x)$$

$$x \ll_{M,\varphi} y \text{ if and only if } \varphi(x - z) \leq \varphi(z) \vee \varphi(y - z) \text{ for every } z \in E.$$

For $\varphi = \|\ \|$, the norm of E, we have $\ll_{L,\varphi} = \ll_L$ and $\ll_{M,\varphi} = \ll_M$. For E a Banach lattice and $\varphi = |\ |$, the modulus on E, we have $\ll_{L,\varphi} = \ll_{M,\varphi} = \ll_o$ an thus studying $\ll_{L,\varphi}$ or $\ll_{M,\varphi}$ is a way to unify results from Banach lattice theory and isometric theory of Banach spaces.

The fact that $\ll_{M,\varphi}$ satisfies the condition AE4) above can be argued as follows: AE4) has a local character i.e., we may restrict ourselves to the case where dim $E < \infty$. Then there exists $x \in X$, $x > 0$, such that $\varphi(E) \subset X_x = \{y \mid |y| \leq \alpha x$ for a suitable $\alpha > 0\}$. We shall consider on X_x the norm

$$\|y\|_x = \inf\{\alpha \mid |y| \leq \alpha x\}.$$

Then a classical result due to Kakutani-Krein asserts that $(X_x, \|\ \|)$ is a Banach lattice-algebraic, isometric and lattice isomorphic to a space $C(S,\mathbb{R})$ for a suitable compact Hausdorff space S. Moreover The inclusion $i_x : X_x \to X$ is a continuous lattice morhism. So we are led to the case where X is a space $C(S,\mathbb{R})$. In that case,

$$\begin{aligned} x \ll_{M,\varphi} y &\text{ if and only if } \varphi(x - z)(s) \leq \varphi(z)(s) \vee \varphi(y - z)(s) \\ &\text{ for every } z \in C(S,\mathbb{R}) \text{ and every } s \in S; \\ &\text{ if and only if } x \ll_M y \text{ with respect to every} \\ &\text{ seminorm } p_s(.) = \varphi(.)(s),\ s \in S \end{aligned}$$

and the proof of AE4) reduces to the case of $\ll_M$.

In what follows we shall be concerned with the duality of vector norms. Since we cannot give a satisfactory reference for that subject we shall give the details here.

2.1 *Definition* (L.V. Kantorovich). A vector norm $\varphi : E \to X$ is said to have

the Riesz decomposition property (abbreviated, **RDP**) provided that for every $u \in E$ and every $x_1, x_2 \in X_+$ with $\varphi(u) \leq x_1 + x_2$ there are $u_1, u_2 \in E$ such that $u = u_1 + u_2$ and $\varphi(u_1) \leq x_1$, $\varphi(u_2) \leq x_2$.

If a vector norm φ has *RDP* then φ is *fully valued* i.e.,

$$0 \leq x \leq \varphi(u) \text{ implies } x = \varphi(v) \text{ for a suitable } v \in E.$$

Clearly, not every vector norm is fully valued. Any "scalar" norm, as well as the modulus of a Banach lattice, has *RDP*. In both cases the vector norm is isometric. Other examples of isometric vector norms with *RDP* are indicated below. We shall need the following technical lemma:

2.2 LEMMA. *Let E be a vector space, X an order complete vector lattice, $U : E \to X$ a linear map and $P_1, P_2 : E \to X$ sublinear maps such that $U(u) \leq P_1(u) + P_2(u)$ for every $u \in E$. Then there exist linear maps $U_1, U_2 : E \to X$ such that $U = U_1 + U_2$ and $U_i(u) \leq P_i(u)$ for every $u \in E$, $i \in \{1,2\}$.*

Proof. Consider the sublinear map $P : E \times E \to X$ given by $P(u_1,u_2) = P_1(u_1) + P_2(u_2)$. Let $\Delta = \{(u,u) \mid u \in E\}$. The map $V : \Delta \to X$ given by $V(u,u) = U(u)$ satisfies the inequality $V(u,u) \leq P(u,u)$ for every $(u,u) \in \Delta$. Consequently, the operatorial version of Hahn-Banach theorem allows us to extend V to a linear map $V : E \times E \to X$ such that $V(u_1,u_2) \leq P(u_1,u_2)$ for every $(u_1,u_2) \in E \times E$. The maps $U_1(u) = V(u,0)$, $U_2(u) = V(0,u)$ have all required properties. ∎

2.3 *Examples.* I am indebted to Dan T. Vuza for the following examples of isometric vector norms with *RDP*.

i) Let E be a Banach space and let X be an order complete vector lattice. A linear operator $U : E \to X$ is called **majorizing** if U maps the unit ball $\overline{B}_1(E)$ of E into an order bounded subset of X. The set of all majorizing operators from E into X is a vector space denoted by $M(E, X)$. The map $\mu : M(E, X) \to X$ given by $\mu(U) = \sup U(\overline{B}_1(E))$ is a vector norm having *RDP*. To see this, let $\mu(U) \leq x_1 + x_2$. Lemma 2.2 applied for the linear map U and the sublinear maps $P_1, P_2 : E \to X$ given by $P_i(u) = \|u\| \cdot x_i$ ($i \in \{1,2\}$) yields the linear maps U_1, U_2 such that $U = U_1 + U_2$ and $U_i(u) \leq \|u\| \cdot x_i$ for every $u \in E$, $i \in \{1,2\}$. It follows that $u_i \in M(E,X)$ and $\mu(u_i) \leq x_i$ for $i \in \{1,2\}$.

Suppose now that X is a Banach lattice and define the norm $\| \ \|_M$ on $M(E,X)$ by $\|U\|_M = \|\mu(U)\|$. Endowed with this norm, $M(E,X)$ becomes a Banach space and μ becomes an isometric vector norm.

ii) Let E be a Banach space and let X be a vector lattice. A linear operator $U : X \to E$ is called **cone summable** if for every $x \in X_+$ we have

$$\sigma(U)x = \sup\left\{ \sum_{i=1}^{n} \left\| U(x_i) \right\| \,\middle|\, n \in \mathbb{N}^*, x_i \in X_+, \sum_{i=1}^{n} x_i = x \right\} < \infty.$$

The set of all cone summable operators from X into E is a vector space, denoted by $S_+(X,E)$. The map $x \to \sigma(U)x$ can be extended by linearity to a positive linear form on X, denoted by $\sigma(U)$. Thus we obtain a vector norm $\sigma : S_+(X,E) \to X^{\sim}$, where $X^{\sim}$

denotes the vector lattice of all order bounded linear forms on X.

If E is a dual Banach space then σ has *RDP*. This can be shown as follows: Let F be a predual of E i.e., a Banach space F such that $F' = E$. We associate to every $U \in S_+(X,E)$ a map $\overline{U} \in M(F, X^\sim)$ given by $\overline{U}(v)(x) = U(x)(v)$. The correspondence $U \to \overline{U}$ establishes a bijection between $S_+(X,E)$ and $M(F,X^\sim)$ such that $\mu(\overline{U}) = \sigma(U)$; it remains to use the example i) above in order to conclude the proof.

Suppose now that X is a Banach lattice and define the norm $\| \, \|_S$ on $S_+(X,E)$ by $\|U\|_S = \|\sigma(U)\|$. Endowed with this norm $S_+(X,E)$ becomes a Banach space and σ becomes an isometric vector norm.

iii) Let E be a Banach space and let X be a Banach lattice. We denote by i_E the canonical inclusion of E into E'' i.e.,

$$i_E(x)(x') = x'(x), \; x \in E, x' \in E'.$$

Let $M_*(E',X)$ be the space of all linear operators $U : E' \to X$ satisfying the following requirements:

a) $U'(X') \subset i_E(E)$.

b) There exists an $x \in X$ such that $U(\overline{B}_1(E'))$ is contained and totally bounded in $(X_x, \| \, \|_x)$.

It is easy to show that the supremum of a totally bounded set of a Banach lattice with a strong order unit always exists. In fact, via Kakutani-Krein representation theorem every such a space is a space $C(S,\mathbb{R})$. Consequently, for every $U \in M_*(E',X)$ it makes sense $\mu(U) = \sup U(\overline{B}_1(E'))$ in X. The map $\mu : U \to \mu(U)$ is a vector norm on $M_*(E',X)$. With respect to the norm $\| \, \|_M$ given by $\|U\|_M = \|\mu(U)\|$, $M_*(E',X)$ becomes a Banach space and μ becomes an isometric vector norm.

The vector norm μ has *RDP*. Indeed, because every order ideal $(X_x, \| \, \|_x)$ is algebraic, lattice and isometric isomorphic to a space $C(S,\mathbb{R})$, it suffices to prove the assertion in the case where X is itself a space $C(S,\mathbb{R})$. In this case, for every $U \in M_*(E',C(S,\mathbb{R}))$ there exists a continuous map $F : S \to E$ such that $U(u)(s) = u(F(s))$ for every $u \in E'$ and $s \in S$. The fact that $\mu(U) \le x_1 + x_2$ means $\|F(s)\| \le x_1(s) + x_2(s)$ for every $s \in S$. Consider the continuous maps $F_i : S \to E$ $(i \in \{1,2\})$ given by

$$F_i(s) = (x_1(s) + x_2(s))^{-1} \cdot x_i(s) \cdot F(s), \text{ if } x_1(s) + x_2(s) > 0$$

$$F_i(s) = 0 \qquad , \text{ if } x_1(s) + x_2(s) = 0.$$

The operators $U_i \in M_*(E',C(S,\mathbb{R}))$ given by $U_i(u)(s) = u(F_i(s))$ $(i \in \{1,2\})$, satisfy all requirements in the definition of Riesz decomposition property.

The Banach space $M_*(E',X)$ is isometric to the M-tensor product.

The interest for vector norms with *RDP* is justified by the possibility of dualizing such norms. Indeed, given an isometric vector norm $\varphi : E \to X$ with *RDP* the dual vector norm $\varphi' : E' \to X'$ of φ is defined by

$$\varphi'(u')(x) = \sup_{\varphi(u) \leq x} |u'(u)|$$

for every $u' \in E'$ and $x \in X_+$; the map $\varphi'(u') : X_+ \to \mathbb{R}_+$ is positively homogeneous and additive (because of *RDP*) and thus extends uniquely to a positive linear form on X, also denoted by $\varphi'(u')$. It is clear that φ' is a vector norm.

2.4 PROPOSITION. *φ' is an isometric norm with RDP.*

Proof. The fact that φ' is isometric is a straightforward calculation. In fact,

$$\varphi'(u') = \sup_{\substack{\|x\| \leq 1 \\ x \geq 0}} \varphi'(u')(x) = \sup_{\substack{\|x\| \leq 1 \\ x \geq 0}} \sup_{\varphi(u) \leq x} |u'(u)| =$$

$$= \sup_{\|u\| \leq 1} |u'(u)| = \|u'\|.$$

To see that φ' has *RDP* let $\varphi'(u') \leq x'_1 + x'_2$. Equivalently,

$$u'(u) \leq x'_1(\varphi(u)) + x'_2(\varphi(u)),\ u \in E.$$

By applying Lemma 2.2 to the linear form u' and the sublinear forms $u \to x'_i(\varphi(u))$ ($i \in \{[1,2\}$), we obtain the linear forms u'_i such that $u'_i(u) \leq x'_i(\varphi(u))$ (for $u \in E$ and $i \in \{1,2\}$) and $u' = u'_1 + u'_2$. Consequently $\varphi'(u'_i) \leq x'_i$ and the proof is done. ■

Proposition 2.4 allows us to consider φ'', φ''' and so on.

2.5. PROPOSITION. *Let $\varphi : E \to X$ an isometric vector norm with RDP. Then*

$$x'(\varphi(u)) = \sup_{\varphi'(u') \leq x'} |u'(u)|$$

for every $u \in E$ and $x' \in X'_+$. In other words, $\varphi''(i_E(u)) = i_x(\varphi(u))$.

Proof. The map $u \to x'(\varphi(u))$ is a seminorm on E and the set of all linear forms majorated by it is $\{u' \mid u' \in E',\ \varphi'(u') \leq x'\}$; thus our assertion is a consequence of Hahn - Banach extension theorem.

The duals of the $\mathbb{R}$ - valued norms are the usual dual norms.

The dual of the vector norm $u \to |u|$ on a Banach lattice E is the vector norm $u' \to |u'|$ on E'.

The dual of $M_*(E',F)$ can be isometrically identified with $S_+(F,E')$. When this identification is performed, one can show, by using the techniques in [Ch], that the dual norm of the vector norm μ on $M_*(E',F)$ is the vector norm σ on $S_+(F,E')$.

3. BANACH SPACES HAVING AN ORDER CONTINUOUS NORM

Throughout this section E will denote a Banach space endowed with an AE-order relation $\ll$.

3.1 *Definition.* We shall say that the norm of E is ($\ll$ -) **order continuous** provided that every downwards directed net $(x_a)_a$ of elements of E is norm convergent.

If the norm of E is order continuous, then every upwards directed $\ll$ - majorized net of elements of E is also norm convergent. In fact, if $(x_a)_a$ is such a net and $x_a \ll y$ for every a then the net $(y-x_a)_a$ is $\ll$ - decresing. The argument is as follows: $x_a \ll x_b \ll y$ implies $y - x_b \ll y$ and $x_b - x_a \ll x_b \ll y$ and thus by condition AE4) we conclude that $y - x_b \ll (y - x_b) + (x_b - x_a) = y - x_a$.

Also, in Definition 3.1 above it suffices to deal with sequences instead of nets.

For $u,v \in E$ with $u \ll v$, we define the ($\ll$ -) **order interval** of extremities u and v as the set

$$[u,v] = \{x \mid x \in E, u \ll x \ll v\}.$$

In order to underline the order relation under study we shall use also notation like $[u,v]_\ll$, $[u,v]_L$ (when $\ll = \ll_L$) and so on. Particularly, if E is a Banach lattice we must distinguish carefully among the intervals $[u,v]_\ll$ and the usual order intervals, denoted by $[u,v]$.

3.2 Lemma. *If $u \ll v$ then* $[u,v] = \{x \mid x \in E, x - u \ll v - u\}$.

Proof. In fact, if $u \ll x \ll v$ then $x - u \ll x \ll v$ and $v - x \ll v$, so by AE4) we infer that $x - u \ll (v - x) + (x - u) = v - u$. Conversely, from $x - u \ll v - u \ll v$ and $u \ll u \ll v$ we infer that $x - u + u \ll v - u + u$ i.e., $x \ll v$. Also, from $x - u \ll x - u \ll v$ and $u \ll v$ we infer that $u \ll u + (x - u) = x \ll u + v - u = v$. ■

If E is a Banach lattice, then from Lemmata 1.10 and 3.2 above we infer that

$$[u,v]_\circ = [u \wedge v, u \vee v]$$

and thus $[-u,u]_\circ = [-u,u]$ for $u \geq 0$.

3.3 Proposition. *Every $\ll$ - order interval $[u,v]$ is convex, bounded and norm closed.*

Proof. Let $x,y \in [u,v]$ and $\alpha \in [0,1]$. Then by AE2), $\alpha u \ll x \ll \alpha v$ and $(1 - \alpha)u \ll (1 - \alpha)y \ll (1 - \alpha)v$ which yields, via AE4), that $u \ll \alpha x + (1 - \alpha)y \ll v$. By Proposition 1.2 iii) above, $u,v \in \overline{B}_{\|v\|}(0)$ and thus $[u,v]$ is a bounded set. The fact that $[u,v]$ is norm closed follows from Lemma 3.2 and AE6). ■

3.4 Corollary. *If E has $\ll$ - order continuous norm and $(x_a)_a$ is a $\ll$ - downwards directed net of elements of E then $(x_a)_a$ is norm convergent to its $\ll$ - g.l.b.*

Proof. Suppose that $\|x_a - x\| \to 0$. Since all intervals $[0,x_a]$ are norm closed, it follows that $x \ll x_a$ for every a. The same argument shows that if $y \ll x_a$ for every a then $y \ll x$. ■

For E a Banach lattice and $\ll = \ll_o$, Definition 3.1 above agrees with the usual concept of order continuity (called here (o) - **continuity**) as known in Banach lattice theory,

$$0 \le x_a \downarrow \text{ in } E \text{ implies } (x_a)_a \text{ is norm convergent.}$$

See [LTz], p.7, or [S], p.92. Most of the results in this section are inspired by this particular case.

It was noticed in [AE], p.107, that the norm of every Banach space is $\ll_L$ - continuous. We shall prove here a more general result.

3.5 PROPOSITION. *Let X be a Banach lattice with (o) - continuous norm and let $\varphi : E \to X$ an isometric vector norm. Then the norm of E is $\ll_{L,\varphi}$ - continuous.*

Proof. Suppose that $(x_a)_a$ is a $\ll$ - downwards directed net of elements of E. Then $(\varphi(x_a))_a$ is a downwards directed net of positive elements of X and thus norm convergent. Since for $c \ge a, b$ we have

$$\varphi(x_a - x_b) \le \varphi(x_a - x_c) + \varphi(x_b - x_c) = \\ = \varphi(x_a) + \varphi(x_b) - 2\varphi(x_c)$$

it follows that $(x_a)_a$ is a Cauchy sequence in the Banach space E. ■

Another criterion of order continuity is as follows:

3.6 PROPOSITION. *Suppose that all order intervals $[u,v]$ of E are weakly compact. Then the norm of E is order continuous.*

As in the case of Banach lattices, this is an immediate consequence of

3.7 DINI'S LEMMA. *Suppose that $(x_\alpha)_\alpha$ is a downwards directed net of elements of E, weakly convergent to x. Then $\|x_\alpha - x\| \to 0$.*

Proof. Notice first that $x \ll x_\alpha$ for all α. In fact, all the order intervals $[0, x_\alpha]$ are convex and norm closed, which implies that they are also weakly closed. Since $(x_\alpha)_\alpha$ is downwards directed it follows that $x \in [0, x_\alpha]$ for every α.

The net $(x_\alpha - x)_\alpha$ is also downwards directed. In fact, if $x_\alpha \ll x_\beta$ then $x_\beta - x_\alpha \ll x_\beta$ and $x_\alpha - x \ll x_\alpha \ll x_\beta$, so by AE4) we infer that $x_\alpha - x \ll (x_\alpha - x) + (x_\beta - x_\alpha) = x_\beta - x$.

Since $x_\alpha - x \xrightarrow{w} 0$, for every $\varepsilon \to 0$ there exists a convex combination

$$\sum_{k=1}^{N} \lambda_k \left(x_{\alpha(k)} - x\right)$$

of norm $\le \varepsilon$. A new appeal to AE4) shows that for $\alpha \ge \alpha(1), \ldots, \alpha(N)$ sufficiently large,

$$x_\alpha - x = \sum_{k=1}^{N} \lambda_k \left(x_\alpha - x\right) \ll \sum_{k=1}^{N} \lambda_k \left(x_{\alpha(k)} - x\right)$$

and thus $\|x_\alpha - x\| \le \varepsilon$, by Proposition 1.2 iii). ■

By Proposition 3.6, the norm of every reflexive Banach space is order continuous regardless of *AE*-order relation we consider on it.

A result due independently to H.P. Lotz [L] and Niculescu [N1] asserts that the converse of Proposition 3.6 is valid for E a Banach lattice and $\ll = \ll_o$. However this is not longer true in the general setting e.g., consider the case where $E = \mathcal{o}$, $\ll = \ll_L$ and x is the unit of $\mathcal{o}$.

We come now to the main source of Banach spaces with order continuous norm. Our construction exploits the fact that every von Neumann algebra has a unique (up to isometry) predual and its real part is an ordered Banach space with a strong order unit. See [SZ].

Given a topology τ on a Banach space E, we shall denote by τo the τ - operator topology on $L(E,E)$ i.e.,

$$T_\alpha \xrightarrow{\tau o} T \text{ if and only if } T_\alpha(x) \to T(x) \text{ for every } x.$$

3.8 LEMMA. *Let $\mathcal{A}$ be a commutative von Neumann algebra included by $L(E,E)$, such that the inclusion $\mathcal{A} \to L(E,E)$ is a morphism of unital normed algebras mapping w' - convergent nets into τo - convergent nets, where $\tau = \sigma(E, \mathcal{X})$ is the weak topology associated to a certain separating subset $\mathcal{X}$ of E'.*
Then

$$x \ll_{\mathcal{A}} y \textit{ if and only if } x = Uy \textit{ for a suitable } U \in \mathcal{A},\ 0 \le U \le i$$

is an AE-order relation on E satisfying the following two conditions:

i) *Every $\ll_{\mathcal{A}}$ - downwards directed net $(x_\alpha)_\alpha$ of elements of E is τ - convergent to its* g.l.b.

ii) *All $\ll_{\mathcal{A}}$ - order intervals* $[u,v]$ *are τ - closed.*

Proof. We shall put $[0,I] = \{U \mid U \in \mathcal{A}, 0 \le U \le I\}$. Because $\mathcal{A}$ is space $L^\infty(\mu)$, it satisfies the following Radon-Nikodym type property

(*) For every $S, T \in \mathcal{A}_+$ there exists a $U \in [0,I]$ such that $S = U(S + T)$.

It is immediate that $\ll_{\mathcal{A}}$ is a reflexive transitive relation on E satisfying the conditions AE1)-AE3) and AE5) in Definition 1.1.

For the antisimmetry of $\ll_{\mathcal{A}}$ suppose that $x \ll_{\mathcal{A}} y$ and $y \ll_{\mathcal{A}} x$. Then there are $S,T,U \in [0,I]$ such that

$$x = Sy,\ y = Tx \text{ and } I - S + U(I - ST).$$

The existence of U is guaranteed by (*). We have $y - x = (I - S)y = U(I - S + S(I - T))y = U(0) = 0$ i.e., $x = y$.

For AE4), suppose that $x_1 \ll_{\mathcal{A}} y_1$, $x_2 \ll_{\mathcal{A}} y_2$ and $y_1 \ll_{\mathcal{A}} y_1 + y_2$ i.e., $x_1 = S_1 y_1$, $x_2 = S_2 y_2$, $y_1 = T(y_1 + y_2)$ for suitable S_1, S_2, $T \in [0,I]$. Then $x_1 = S_1 T(y_1 + y_2)$, $x_2 = S_2(I - T)(y_1 + y_2)$ and $0 \le S_1 T + S_2(I - T) \le I$, which implies $x_1 \parallel x_2$ and $x_1 + x_2 \ll_{\mathcal{A}} y_1 + y_2$.

For AE6), let $x_\alpha \ll_{\mathcal{A}} y$ $(\alpha \in A)$ with $\|x_\alpha - x\| \to 0$. Then for each a there exists a $U_\alpha \in [0,I]$ such that $x_\alpha = U_\alpha y$. Since the unit ball of $\mathcal{A}$ is w' - compact, we may assume in addition that $(U_\alpha)_\alpha$ is w' - convergent to a $U \in [0,I]$. Then $x'(U_\alpha y) \to x'(Uy)$ for every $x' \in \mathcal{X}$ and thus $x = Uy$. This ends the proof that $\ll_{\mathcal{A}}$ is an *AE*-order relation on E.

Let $(x_\alpha)_\alpha \subset [u,v]$ a net τ - convergent to x. Since $x_\alpha = U_\alpha v$ for suitable $U_\alpha \in [0,I]$ and the unit ball of $\mathcal{A}$ is w' - compact, we may assume in addition that $(U_\alpha)_\alpha$ is w' - convergent to a $U \in [0,I]$. Then $x = Uv$ and thus $x \ll_{\mathscr{A}} v$. For the inequality $u \ll_{\mathscr{A}} x$, notice that $x_\alpha - u \ll_{\mathscr{A}} v$ and repeat the argument above.

Finally let $(x_\alpha)_\alpha$ be a downwards directed net of elements of E such that $x_\alpha \ll_{\mathscr{A}} v$ for every α. We can show as above that x is a τ - cluster point of $(x_\alpha)_\alpha$. Since $(x_\alpha)_\alpha$ is downwards directed and the order intervals of E are τ - closed, it follows that x is the g.l.b. of $(x_\alpha)_\alpha$ and thus the only τ - cluster point of $(x_\alpha)_\alpha$. ■

From Lemmata 3.7 and 3.8 we infer that the norm of E is $\ll_{\mathscr{A}}$- order continuous provided that τ is the weak topology on E.

4. FACIAL CONES

The facial structure (of the unit ball) of a Banach space E will be expressed in terms of cones. By a **cone** we shall mean any non-empty subset C of E such that $C = \bigcup_{\alpha \geq 0} \alpha C$. A cone C of E is said to be **proper** (respectively **hereditary** with respect to an AE-order relation $\ll$ on E) provided that $C \cap (-C) = \{0\}$ (respectively $x \ll y$ and $y \in C$ implies $x \in C$). A cone C is said to be **convex** provided that $C + C \subset C$.

A convex cone C of E is said to be ($\ll$-) **facial** provided that the following two conditions are satisfied:

FC1) C is hereditary;

FC2) $x \parallel y$ for every x and y in C;

By F 2) and anti-symmetry of $\ll$, every facial cone is proper.

For every $x \in E$, the convex cone

$$C(x) = \{y \mid y \in E, y \ll \alpha x \text{ for a suitable } \alpha \geq 0\}$$

is the smallest facial cone containing x i.e., **the facial cone generated by** $\boldsymbol{x}$.

It is easily seen that a convex cone C of E is facial if and only if $C = \bigcup_{x \geq C} C(x)$ and $x_1, x_2 \in C$ implies $C(x_1) + C(x_2) \subset C(x_1 + x_2)$.

A facial cone may not be closed e.g., see the case where $E = L^2[0,1]$, $\ll = \ll_o$ and $C = C(1)$; in this case $\overline{C} = E_+$.

The facial picture of a Banach space is clarified by the following results.

4.1 LEMMA. *If C_1 and C_2 are facial cones such that $C_1 \subset C_2$ and $C_1 \neq C_2$ then $C_1 \subset \operatorname{Fr} C_2$.*

Proof. Suppose that the contrary is true. Then would exist an $x \in C_2$ and an $r > 0$ such that $B_r(x) \cap C_2 \subset C_1$. If $z \in C_2 \setminus C_1$, $\|z\| < r$ then $x + z \in B_r(x) \cap C_2 \subset C_1$. Since $x, z \in C_2$, we have $z \ll z + x$. Since C_1 is hereditary, the later inequality implies that $z \in C_1$, a contradiction. ■

4.2 COROLLARY. *Let C be a facial cone and $x \in C \setminus \operatorname{Fr} C$. Then $C = C(x)$.*

An equivalent way to describe the facial picture of a Banach space is indicated below.

4.3 *Definition*. By a **facial structure** on a Banach space E we shall mean any family $(C_x)_{x \in E}$ of proper convex cones of E, satisfying the following conditions:

i) $x \in C_x$ and $\alpha C_x = C_{\alpha x}$ for every $x \in E$ and $\alpha \in \mathbb{K}$;

ii) If $y, z \in C_x$ then $C_y + C_z \subset C_{y+z}$;

iii) $y \in C_x$ implies $C_y \subset C_x$;

iv) For every $x \in E$ the set $[0,x] = \{y \mid y, x - y \in C_x\}$ is convex, closed and contained in the ball $\overline{B}_{\|x\|/2}(x/2)$.

The facial structure associated to an AE-order relation « is $(C(x))_{x \in E}$. Every facial structure on E can be obtained in such a way. In fact, if $(C_x)_{x \in E}$ is a facial structure on E then by letting

$$x \ll y \text{ if and only if } x, y - x \in C_y$$

we obtain an AE-order relation « on E such that $C_y = \{x \mid x \ll \alpha y \text{ for a suitable } \alpha \geq 0\}$ for every $y \in E$. Consequently *the AE-order relations and the facial structures on a given Banach space are in a natural one-to-one correspondence*.

By Lemma 4.1, every facial structure gives rise to a certain partition of the whole space into simmetrical cones.

The facial cones allow us to develop an ideal theory that is in many respects comparable with that in Banach lattice theory. For, we need an observation, important for itself.

To any convex proper cone C of vector space E we can associate an ordering on E, compatible with the linear structure:

$$x \leq y \ (\text{mod } C) \text{ if and only if } y - x \in C.$$

4.4 LEMMA. *If E is endowed with an AE-order relation « and x and y are two elements of E then the following assertions are equivalent*:

i) $x \ll y$;

ii) $0 \ll x \ll y$ (mod C) *for a suitable facial cone C containing y*;

iii) $0 \ll x \ll y$ (mod C) *for every facial cone C containing y*.

Proof. i) $\Rightarrow$ iii). If $x \ll y$ and C is a facial cone containing y then x, y-$x \in C$ because C is hereditary. Clearly, iii) $\Rightarrow$ ii).

ii) $\Rightarrow$ i). By hypotheses, x and $y - x$ are in C. By FC2), $x \parallel y$-x and thus $x \ll y = x + (y - x)$. ■

The principal ideal generated by an element x of E is defined as the set $E_x = \text{Span}\, C(x)$. The real part of E_x,

$$\text{Re}\, E_x = C(x) - C(x)$$

will be endowed with the ordering associated to $C(x)$ and the norm

$$\|y\|_x = \inf \{\alpha \mid \alpha \in \mathbb{R}_+, \exists\, u \ll \alpha x, v \ll \alpha x, y = u - v\}.$$

The fact that $\|y\|_x = 0$ implies $y = 0$ can be proved as follows. Let $u_n, v_n \in C(x)$ with $y = u_n - v_n$ and $u_n, v_n \ll x/n$ for every $n \in \mathbb{N}^*$. By proposition 1.2 iii), $\|u_n\|, \|v_n\|$ « $\|x\|/n$, so by letting $n \to \infty$ we conclude that $y = 0$.

4.5 LEMMA. *For* $y \in E_x$ *and* $\alpha \geq 0$ *the following assertions are equivalent*:

i) $y = u - v$, *where* $u, v \ll \alpha x$;

ii) $-\alpha x \leq y \leq \alpha x$ (mod $C(x)$);

iii) $y + \alpha x \ll 2\alpha x$.

Proof. Clearly i) $\Rightarrow$ iii) and ii) $\Leftrightarrow$ iii). As concerns the implication iii) $\Rightarrow$ i), notice that $y + \alpha x \ll 2\alpha x$ yields $\alpha x - y \ll 2\alpha x$ and thus $y = (y + \alpha x)/2 - (\alpha x - y)/2$. ■

From Lemma 4.5 and AE5) we infer that

$$E_x = \{y \mid y + \alpha x \ll 2\alpha x \text{ for a suitable } \alpha \geq 0\}$$

and the canonical inclusion $i_x : \mathrm{Re}\, E_x \to E$ is continuous with $\|i_x\| \leq \|x\|$.

4.6 LEMMA. Re E_x *is an ordered Banach space with a strong order unit*, x.

Proof. We have to prove only the completeness of Re E_x. For, let $(y_n)_n$ be a Cauchy sequence in Re E_x. Since i_x is continuous, $(y_n)_n$ is also a Cauchy sequence in E and thus there exists a $y \in E$ such that $\|y_n - y\| \to 0$. On the other hand, for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\|y_m - y_n\|_x \leq \varepsilon \text{ for every } m, n \geq N \text{ i.e.}$$

$$-\varepsilon x \leq y_m - y_n \leq \varepsilon x \text{ (mod } C(x)) \text{ for every } m, n \geq N$$

by Lemma 4.5 above. By letting $m \to \infty$ we infer that

$$-\varepsilon x \leq y - y_n \leq \varepsilon x \text{ (mod } C(x)) \text{ for every } n \geq N$$

which yields $y \in E_x$ and $\|y - y_n\|_x \leq \varepsilon$ for $n \geq N$. ■

The following proposition combines classical results due to Kadison Kakutani and Krein.

4.7 PROPOSITION. i) Re E_x *is algebraic isometric and order isomorphic to the ordered Banach space* $A(K,\mathbb{R})$, *where* K *denotes the* w' - *compact convex set of all states of* Re E_x.

ii) *Suppose in addition that*

1) *either* Re E_x *is endowed with a bilinear multiplication for which* x *is an identity and* $y, z \in$ Re E_x, $y \geq 0$, $z \geq 0$ *implies* $yz \geq 0$; *or*,

2) Re E_x *is a vector lattice with respect to the ordering* mod $C(x)$.

Then Re E_x *is a commutative Banach algebra algebraic, isometric and order isometric to the Banach lattice* $C(S,\mathbb{R})$, *where* S *denotes the* w' - *compact set of all pure states of* Re E_x.

Proof. For i), see [Kad2]; ii 1) follows from Theorem 1.11, while ii 2) needs the classical representation theorem of *AM*-spaces due to Kakutani and Krein. See [S] for details.

5. FACES AND EXTREME POINTS

The facial cones can be also defined as the cones generated by faces. Suppose that E is a Banach space, $K = \overline{B}_1(0)$ is the unit ball of E and S is the unit spere of E.

5.1 *Definition.* A («-) **face** of K is any subset F of S satisfying the following three conditions:

F1) $x \parallel y$ for every $x,y \in F$;

F2) F is absorbant i.e., $y \in K \setminus \{0\}$, $x \in F$ and $y \ll x$ implies $y / \|y\| \in F$;

F3) $x,y \in F$ and $\alpha \in (0,1)$ implies $\dfrac{\alpha x+(1-\alpha)y}{\|\alpha x+(1-\alpha)y\|} \in F$.

The connection with the usual notion of face is explained below.

5.2 LEMMA. *For F a subset of K the following assertions are equivalent*:

i) *F is a proper face in the classical sense i.e., F is a convex subset of S such that $x,y \in F$, $\alpha \in (0,1)$ and $\alpha x + (1 - \alpha)y \in F$ implies $x,y \in F$.*

ii) *F is a $\ll_L$ -face.*

Proof. i) $\Rightarrow$ ii). Suppose that x and y are two points of F. Since F is convex, $x/2 + y/2 \in F$. Then

$$1=\left\|\frac{x+y}{2}\right\|=\frac{1}{2}\|x\|+\frac{1}{2}\|y\|$$

which implies that $x \parallel y$. For F2), let $y \in K \setminus \{0\}$ and $x \in F$ with $y \ll_L x$. Then $1 = \|x\| = \|y\| + \|x - y\|$ and

$$x=\|y\|\cdot\frac{y}{\|y\|}+\|x-y\|\cdot\frac{x-y}{\|x+y\|}$$

which implies that $y / \|y\| \in F$. The condition F3) is clear.

ii) $\Rightarrow$ i). We shall show first that F is convex. For let $x,y \in F \subset S$ and let $\alpha \in (0,1)$. Since $x \parallel y$, by Proposition 1.2 iv) above we infer that $\|\alpha x + (1 - \alpha)y\| = \|\alpha x\| + \|(1 - \alpha)y\| = 1$ and the desired conclusion follows now from F3).

If $x,y \in K$, $\alpha \in (0,1)$ and $\alpha x + (1 - \alpha)y \in F$ then $1 = \|\alpha x + (1 - \alpha)y\| = \|\alpha x\| + \|(1 - \alpha)y\| \leq 1$, which yields $\|x\| = \|y\| = 1$ and αx, $(1 - \alpha)y \ll_L \alpha x + (1 - \alpha)y \in F$. It remains to apply F2) in order to conclude that x and y are in F. ■

There exists a natural one-to-one correspondence between the (closed) facial cones and the (closed) faces. In fact, if C is a (closed) facial cone of E and $C \neq 0$ then $F = C \cap S$ is a (closed) face and $C = \operatorname{cone} F$, the cone generated by F. The empty set is the face corresponding to the cone $\{0\}$. Conversely, if F is a (closed) non-empty face of K, then $C = \operatorname{cone} F$ is a (closed) facial cone such that $F = C \cap S$ and $C \neq \{0\}$. In fact, suppose that F is closed and let $(x_n)_n$ be a sequence of elements of C such that $\|x_n - x\| \to 0$. If $x \neq 0$ then $\|x_n\| \to \|x\|$ and $x_n/\|x_n\| \to x/\|x\|$. Since F is closed, $x/\|x\| \in F$ and thus $x \in C$.

By Zorn's Lemma, every face is contained in a maximal face. We do not know whether the closure of a face is still a face. This is true for usual faces, so in this case maximal faces are closed.

The following example shows that F1) and F2) above do not yield F3). For, consider the Banach space $E = \ell^\infty(2,\mathbb{R})$, endowed with the *AE*-order relation $\ll_L$. The vertices $u = (-1,1)$ and $v=(1,1)$ of the unit ball K of E belong to the same $\ll_L$ - face of K. Consequently the set $F = \{u,v\}$ satisfies the conditions F1) & F2). Since $\left(\frac{1}{2}u+\frac{1}{2}v\right)/\left\|\frac{1}{2}u+\frac{1}{2}v\right\| \notin F$, F does not satisfies F3).

The case where $E = \ell^\infty(2,\mathbb{R})$, $\ll = \ll_o$ and $F = E_+ \cap S$ shown that a $\ll$ - face is not necessarily convex.

In operator algebra theory it is known the notion of a face of a C^* - algebra. *A face of the C^* - algebra $\mathscr{A}$ is* any convex cone C contained in $\mathscr{A}_+$ such that

$$y\mathscr{A}_+,\ x \in C \text{ and } y \le x \text{ implies } y \in C.$$

In our terminology, the faces of $\mathscr{A}$ are precisely the $\ll_o$ - facial cones of Re $\mathscr{A}_+$, contained in $\mathscr{A}_+$.

Every point x of the unit spere of a Banach space E belongs to a certain face. In fact, face $\{x\} = C(x) \cap S$ is a face, precisely the smallest face containing x.

5.3 *Definition.* A norm 1 element x of E will be called ($\ll$ -) extremal for K provided that $C(x) = \mathbb{R}_+ \cdot x$ i.e., face$\{x\} = \{x\}$.

Since K is the only subset of E whose extreme points are investigated we shall denote by Ex E (or $\mathrm{Ex}_\ll E$) the subset of all extreme points of K. Also, in order to avoid sub-scripts, we shall use notation like $\mathrm{Ex}_L E$ when $\ll = \ll_L$ etc.

Notice that if F is a $\ll$ - hereditary closed subspace of E then

$$\mathrm{Ex}_\ll F = (\mathrm{Ex}_\ll E) \cap F.$$

For $\ll = \ll_L$ we retrieve the classical notion of an extreme point. See Lemma 5.2 above.

5.4 LEMMA. *Let H be a Hilbert space and $\mathscr{A}$ the von Neumann subalgebra of $L(H,H)$ generated by a self-adjoint operator $A \in L(H,H)$. Then* $\mathrm{Ex}_{\mathscr{A}} H$ *consists of all normalized eigenvectors of A.*

Proof. Let $v \in \mathrm{Ex}_{\mathscr{A}}\ H$. Because $0 \le A^-, A^+ \le \|A\| \cdot I$ and $A^-, A^+ \in \mathscr{A}$, it follows that A^-v , $A^+v \ll \|A\| \cdot v$ and thus $Av = A^+v - A^-v = \alpha v$ for a suitable $\alpha \in \mathbb{R}$.

Conversely, let $Av = \alpha v$ with $\|v\| = 1$ and $\alpha \in \mathbb{R}$. Then $f(A)v = f(\alpha) \cdot v$ for every $f \in C(\sigma(A),\mathbb{C})$ i.e., v is an eigenvector for every operator in the C^* - algebra $C^*\ \{A,I\}$, generated by A and I. Since $\mathscr{A}$ is the *wo* - closure of

$C^*\{A,I\}$, then the same is true for every operator in $\mathcal{A}$. Consequently, $x \ll_{\mathcal{A}} v$ implies $x = \lambda\, v$ for a suitable $\lambda \in \mathbb{C}$ i.e., $v \in \mathrm{Ex}_{\mathcal{A}} H$. ■

The notion of an extreme point is very closed to that of discrete element.

5.5 *Definition.* By a ($\ll$ -) **discrete element** of E we shall mean any element x of E such that $u,v \ll x$ and $C(u) \cap C(v) = \{0\}$ implies either u or v is 0.

Clearly, every element of Ex E is discrete. Conversely, every normalized discrete element is also an extreme point. Before indicating the details, we shall notice the particular case of Banach lattices:

5.6 PROPOSITION. *Let E be a Banach lattice.*

i) *An element x of E is* $\ll_o$ *- discrete if and only if it is an atom* i.e., $u,v \le |x|$ *and* $u \wedge v = 0$ *implies either u or v is* 0.

ii) $\mathrm{Ex}_o E$ *coincides with the set of all normalized atoms of E.*

Consequently, in the real case,

$$\mathrm{Ex}_L\, c_0 = \varnothing \qquad \mathrm{Ex}_o\, c_0 = \{\pm\{\delta_{m\,n}\}_m \mid n \in \mathbb{N}\}$$

$$\mathrm{Ex}_L\, C[0,1] = \{\pm 1\} \qquad \mathrm{Ex}_o\, C[0,1] = \varnothing$$

$$\mathrm{Ex}_L\, L^2[0,1] = \{x \mid \|x\| = 1\} \qquad \mathrm{Ex}_o\, L^2[0,1] = \varnothing$$

We shall prove that in general every normalized discrete element is an extreme point. Our argument is essentially finite dimensional and depends upon an analogue of the orthogonal decomposition.

5.7 LEMMA. *Let E be a finite dimensional Banach space endowed with an AE-order relation* $\ll$ *and let* $x \in E$, $x \neq 0$.

Then the cone $C(x)$ *is closed and for every* $e \in E$ *there exist elements u and v in* $[0,e]$ *such that* $e = u + v$, $u \in C(x)$ *and* $C(v) \cap C(x) = \{0\}$.

Proof. We shall show first that the cone $C(x)$ is closed. For, let $(y_n)_n$ be a sequence of elements of $C(x)$ such that $\|y_n - y\| \to 0$ in E. Then $y_n \ll \|y_n\|_x \cdot x$ for every n. Since $\dim E < \infty$, the canonical inclusion $i_x : E_x \to E$ is an isomorphism into and thus $y \in E_x$ and $\|y_n - y\|_x \to 0$. Put $M = \sup \|y_n\|_x$. By AE3) and AE6) above, we infer that $y \ll Mx$ i.e., $y \in C(x)$.

As concerns the decomposition part, consider the set $A_e = \{z \mid z \in C(x),\, z \ll e\}$; $0 \in A_e$ and A_e is inductively ordered by $\ll$. In fact, the order interval $[0,e]$ is compact and thus every increasing net of elements of $[0,e]$ is norm convergent to its l.u.b. By Zorn's lemma, A_e must contain at least one maximal element, say u. It remains to prove that $v = e - u$ satisfies $C(v) \cap C(x) = \{0\}$. In fact, if the contrary is true then would exist a $z \in C(x)$ such that $z \neq 0$ and $z \ll v$. Since $z \ll e - u$, $u \ll u$ and $e - u \parallel u$, by AE4) it follows that $z + u \ll e$ and $u \ll z + u$. Since $C(x)$ is convex, $z + u \in C(x)$ and this fact contradicts the maximality of u. Consequently $C(v) \cap C(x) = \{0\}$. ■

5.8 LEMMA. *Let E be a finite dimensional Banach space endowed with an AE-order relation* $\ll$. *Then* Ex E *consists precisely of all normalized discrete elements of* E.

Proof. Suppose that e is a normalized discrete element of E and $e \in \operatorname{Ex} E$. Then there exists an $x \in E$ such that $x \ll e$ and $x \notin \mathbb{R}_+ \cdot e$. Put $\alpha = \sup \{\lambda \mid \lambda e \ll x\}$. Then $x - \alpha e \neq 0$ and $C(x - \alpha e) \cap \mathbb{R}_+ \cdot e = \{0\}$. In fact, if $\mu e \ll \lambda(x - \alpha e)$ with $\mu, \lambda \in \mathbb{R}_+^*$, then $(\mu + \lambda\alpha)e \ll \lambda x$ i.e., $(\mu/\lambda + \alpha)e \ll x$, in contradiction with the definition of α. By Lemma 5.7, e admits a decomposition

$$e = u + v$$

with $u \in C(x - \alpha e)$ and $C(v) \cap C(x - \alpha e) = \{0\}$. We shall prove that both u and v are different to 0, which will contradict the fact that e is discrete.

If $v = 0$, then $e = u \in C(x - \alpha e)$. Or, $C(x - \alpha e) \cap \mathbb{R}_+ \cdot e = \{0\}$.

If $u = 0$, then $e = v$ and thus $C(e) \cap C(x - \alpha e) = \{0\}$. Or, $x - \alpha e \ll x \ll e$, so that $C(x - \alpha e) \subset C(e)$.

Consequently, e is an extreme point of E. The other implication is clear. ■

5.9 THEOREM. *Let E be a Banach space endowed with an AE-order relation* $\ll$. *Then* Ex E *consists precisely of all normalized discrete elements of* E.

Proof. We have only to prove that every normalized discrete elements e of E is also an extreme point i.e.,

$$C(e) = \mathbb{R}_+ \cdot e .$$

For, notice that $\ll$ induces on every finite dimensional subspace F of E an *AE*-order relation $\ll_F$ given by

$$x \ll_F y \text{ if and only if } x \text{ and } y \text{ belong to } F \text{ and } x \ll y \text{ in } E.$$

By Lemma 5.8, $C(e) \cap F = \mathbb{R}_+ \cdot e$ for every finite dimensional subspace F which contains e and thus $C(e)$ is indeed $\mathbb{R}_+ \cdot e$. ■

It is worthwhile to mention that every extremal point x belonging to a facial cone C generates an extremal ray $\mathbb{R}_+ \cdot x$ of C i.e.,

$$x = \alpha u + (1 - \alpha)v \text{ with } u, v \in C \text{ and } \alpha \in (0,1) \text{ implies } u, v \in \mathbb{R}_+ \cdot x.$$

As shows the following example, not every extremal ray of C is generated by an extreme point. In fact, let $E = L^2[0,1]$, $\ll\, = \,\ll_o$ and $C = C(1)$. $\operatorname{Ex}_o E = \emptyset$ because E has no atoms. However every characteristic function $\chi_A \in E$ gives rise to an extremal ray of C. The same example shows that Krein-Milman Theorem may be not valid for $\ll\, \neq\, \ll_L$.

We shall prove in section 10 an analogue of Hilbert-Schmidt Theorem that brings together several types of finite dimensional decompositions including the orthogonal and the lattice ones. The basic ingredient is the case of finite dimensional spaces.

5.10 THEOREM. *Suppose that* $\dim E = n$ *and E is endowed with an AE-order relation* $\ll$. *Then for each $x \in E$ there exist scalars $\alpha_1, \ldots, \alpha_n \in [0, \|x\|]$ and* $\ll$ -

extreme points $e_1,\dots,e_n \in C(x)$ *such that*

$$x = \alpha_1 e_1 + \dots + \alpha_n e_n .$$

Proof. The assertion is clear for x an extreme point.

Suppose that $x \notin \operatorname{Ex} E$, $\|x\| = 1$. We shall prove first that there exist discrete elements f such that $f \ll x$ and $f \neq 0$. In fact, by Lemma 5.8 above, there exist elements $u, v \in [0,x] \setminus \{0\}$ such that $C(u) \cap C(v) = \{0\}$. By Lemma 4.1, $C(u) \subset \operatorname{Fr} \ C(x)$ and thus $\dim C(u) < \dim C(x) \leq n$. Consequently, in at most n steps we are led to a discrete element f_1 with $f_1 \ll x$ and $f_1 \neq 0$. Put $e_1 = f_1 / \|f_1\|$ and $\alpha_1 = \sup \{\lambda \mid \lambda e_1 \ll x\}$. Then $e_1 \in \operatorname{Ex} E$ and $C(x - \alpha_1 e_1) \cap C(e_1) = C(x - \alpha_1 e_1) \cap \mathbb{R}_+ \cdot e_1 = \{0\}$. If $x - \alpha_1 e_1$ is not discrete the process described above should be continued with $x - \alpha_1 e_1$ instead of x. ■

For $\ll\, = \,\ll_L$, Theorem 5.10 above shows that every x in the unit sphere of E is a convex combination of extreme points. In fact, if

$$x = \alpha_1 e_1 + \dots + \alpha_n e_n$$

with $e_1,\dots,e_n \in C(x)$ and $\alpha_1,\dots,\alpha_n \geq 0$, then, by FC2),

$$1 = \|x\| = \|\alpha_1 e_1\| + \dots + \|\alpha_n e_n\| =$$

$$= \alpha_1 + \dots + \alpha_n.$$

Since $e \in \operatorname{Ex} E \cap C(x)$ if and only if $-e \in \operatorname{Ex} E \cap C(-x)$, Theorem 5.10 includes the classical result due to Caratheodory that states that each point in the unit ball of an n-dimensional Banach space is a convex combination of at most $n+1$ extreme points. In turn, Theorem 5.10 is an easy consequence of Caratheodory's result.

Theorem 5.10 includes also the following result due to Yudin: *Every finite dimensional Banach lattice has a basis formed by atoms* (and thus it is algebraic, topologic and lattice isomorphic to a space $\mathbb{R}^n$).

From Theorem 5.10 we can deduce easily the fact that given an $n \times n$ - dimensional self-adjoint matrix A there exists an orthonormal basis of $\mathbb{C}^n$ formed by eigenvectors of A.

Theorem 5.10 extends to all Banach lattices whose order intervals are compact, asserting the fact that they are discrete. See [Wh]. An open problem in this setting is outlined at the end of this paper.

6. THE DECOMPOSITION DETERMINED BY A CONE

As usually, E will denote a Banach space endowed with an AE-order relation $\ll$, K its unit ball and S the unit sphere of E.

The complementary cone of a cone C of E is defined as the set

$$C^{\perp} = \{x \mid C(x) \cap C = \{0\}\} =$$

$$= \{x \mid y \ll x \text{ and } y \in C \text{ implies } y = 0\}.$$

Notations like $C^{\perp,\ll}$, $C^{\perp,M}$ etc, are intended to underline the order relation under study $\ll$, $\ll_M$ etc.

Generally $C^{\perp}$ is a hereditary cone but $C^{\perp}$ need not be convex or proper even if C has these properties. For example, if E is the Euclidian 2- dimensional space and $\Delta = \{(\alpha,\alpha) \mid \alpha \in \mathbb{R}_+\}$ then

$$\Delta^{\perp,L} = \mathbb{R}^2 \setminus \{(\alpha,\alpha) \mid \alpha \in \mathbb{R}_+^*\}.$$

6.1 THEOREM. *Suppose that E admits weaker locally convex Hausdorff topology τ such that every $\ll$ - upwards directed $\ll$ - majorized net $(x_\alpha)_\alpha$ in E has a τ - convergent subnet.*

Let C be a τ - closed convex cone of E. Then every $x \in E$ admits a decomposition

$$x = u + v$$

where $u \in C$, $u \ll x$ and $v \in C^{\perp}$.

Proof. Let $A = \{y \mid y \in C, y \ll x\}$. Then $0 \in C$ and A is inductively ordered. By Zorn's lemma, A contains at least one maximal element, say u. The $u \in C$, $u \ll x$ and we shall show that $v = x - u \in C^{\perp}$. In fact, if the contrary is true, then would exist a $z \in C$, $z \neq 0$, such that $z \ll v$. Since $z \ll x - u$, $u \ll u$ and $x - u \parallel u$, by AE4) above it follows that $u \ll z + u$ and $z + u \ll x$. Since C is convex, $z + u \in C$ and this fact contradicts the maximality of u. Consequently $v \in C^{\perp}$. ■

The hypotheses of Theorem 6.1 are fulfilled in each of the following two particular cases:

A) E has order continuous norm and τ is the norm topology;

B) E is the dual of the Banach space F (endowed with an isometric vector norm $\varphi : F \to X$ having *RDP*), $\ll = \ll_{M,\varphi'}$ and $\tau = w'$.

The argument in case B) constitutes Corollary 6.3 below.

6.2 LEMMA. *Let $\varphi : E \to X$ be an isometric vector norm with RDP. Then every $\ll_{M,\varphi'}$ - interval $[u', v']$ of E' is w' - compact.*

Proof. By AE4),

$$[u', v'] = [0, v'] \cap (u' + [0, v']),$$

so that it suffices to prove that every order interval $[0, v']$ is w' - compact. For, notice that $[0, v']$ is the intersection of all balls $\overline{B}_{x'}(u') = \{z' \mid z' \in E', \varphi'(z' - u') \leq x'\}$ with $u' \in E'$ and $x' \in X'_+$, that contains 0 and v'. Or, any ball $\overline{B}_{x'}(u')$ is a w' - closed subset of a w' - compact set, $\{z' \mid z' \in E', \|z' - u'\| \leq \|x'\|\}$.

6.3 COROLLARY. *Let E and φ be as above. Then every $\ll_{M,\varphi'}$ - upwards directed $\ll_{M,\varphi'}$ - majorized net $(u'_\alpha)_\alpha$ of elements of E' is w' - convergent to its $\ll_{M,\varphi'}$ - g.l.b.*

Applications of Corollary 6.3 to the study of Boolean algebras of projections will be given in section 9.

The following example outlines the significance of Theorem 6.1 in the case of Banach lattices. Suppose that E is a Banach lattice and $\mathcal{J}$ is a **closed lattice ideal** of E i.e., a closed subspace of E such that $|x| \leq |y|$ and $y \in \mathcal{J}$ implies $x \in \mathcal{J}$. Then the $\ll_0$ - complementary cone of $\mathcal{J}$ is also a closed lattice ideal of E and coincides with the order orthogonal of $\mathcal{I}$,

$$\mathcal{I}^{\perp} = \{x \mid x \in E, |x| \wedge |y| = 0 \text{ for every } y \in \mathcal{I}\}.$$

By Theorem 6.1, if E has (o) - continuous norm then $\mathcal{I}$ gives rise to an orthogonal decomposition $E = \mathcal{I} \oplus \mathcal{I}^{\perp}$ known in Banach lattice theory as **band decomposition.** See [LT2] or [S2].

In what follows we shall be concerned with the geometric consequences of Theorem 6.1.

Let F be a subset of K. The **complementary set** F^c of F is defined as the union of all (« -) faces G contained in (cone $F)^{\perp}$.

6.4 PROPOSITION. *Suppose that the norm E is « - continuous and let F be a closed face of K. Then for every $x \in K$ there exist scalars λ, $\mu \in$ [0,1] and vectors $u \in F$ and $v \in F^c$ such that $x = \lambda u + \mu v$.*

Proof. The cone C generated by F is convex and closed, so by Theorem 6.1 there exist $p \in C$ and $q \in C$ such that $x = p + q$. Since p,q « x and $x \in K$ it follows that $\|p\|$, $\|q\| \leq 1$. If p or q is 0, the proof ends. If $p \neq 0$, and $q \neq 0$, then

$$x = \|p\| \cdot \frac{p}{\|p\|} + \|q\| \cdot \frac{q}{\|q\|}$$

and $u = p / \|p\| \in F$. Since $C(q) \cap C = \{0\}$ face $\{q / \|q\|\} \cap F = \varnothing$ and thus $q / \|q\| \in F^c$. ■

The case where $E = \ell^{\infty}(2, \mathbb{R})$, « = «$_{\alpha}$ and F = face $E_+ \cap S$ shows that the result of Proposition 6.4 is the best possible in the sense that the scalars λ and μ may be arbitrary in [0,1]. On the other hand, in the case where « = «$_L$, Proposition 1.6.4 shows that every $x \in S$ is a convex combination of a point of F and a point of F^c. This was first noticed in [AE], p.108.

7. THE CENTRALIZER

In this section will shall attach to each Banach space E endowed with an AE-order relation « a certain C^* - algebra of operators on E, called the centralizer. The importance of this construction will become clear in the next sections by considering the reversed process.

The idea is to associate to « an order relation on $L(E,E)$ by letting S « T if and only if Sx « Tx for every $x \in E$.

This new order relation satisfies all the conditions in Definition 1.1 above when $L(E,E)$ is endowed with the family of all seminorms $p_x : T \to \|Tx\|$, $x \in E$.

Consider the facial cone generated by I,

$$Z(E)_+ = \{T \mid T \in L(E,E),\ T \text{ « } \alpha I \text{ for a suitable } \alpha \geq 0\}.$$

The centralizer associated to « is defined as the principal ideal generated by I,

$$Z(E) = \operatorname{Span} Z(E)_+$$

Notations like $Z_{«}(E)$, $Z_L(E)_+$ etc, are intended to underline that the order relation under consideration is « respectively «$_L$ etc.

By Lemma 4.5,

$$Z(E) = \{T \mid T \in L(E,E),\ T + \alpha I \ll \beta I \text{ for some } \alpha, \beta \geq 0\}.$$

Since the results in section 4 remain true (with a similar proof) for locally convex spaces, we infer that Re $Z(E)$ is an ordered Banach space with respect to the cone $Z(E)_+$ and the norm $\| \ \|_I$ associated to the strong order unit I. It is also clear that

$$S, T \in Z(E)_+ \text{ implies } ST \in Z(E)_+$$

7.1 PROPOSITION. *Re $Z(E)$ is a commutative Banach algebra algebraic, isometric and lattice isometric to a space $C(\Omega, \mathbb{R})$.*

In the complex case, $Z(E)$ is the complexification of the Re $Z(E)$ and thus it is a complex Banach lattice and also a complex Banach algebra with respect to the modulus

$$|S + iT| = \sup_{0 \leq \theta \leq 2\pi} \left| (\cos\theta) \cdot S + (\sin\theta) \cdot T \right|$$

and the norm

$$\|S + iT\|_I = \| \, |S + iT| \, \|_I$$

where $S, T \in$ Re $Z(E)$.

7.2 THEOREM. i) *$Z(E)$ is a commutative Banach algebra and a Banach lattice that is algebraic, isometric and lattice isometric to a space $C(\Omega, \mathbb{C})$.*

ii) *The norm $\| \ \|_I$ and the operatorial norm coincide on $Z(E)$.*

Proof. The assertion i) follows from Proposition 7.1.

ii) By i), it suffices to prove that $\| \ \|$ and $\| \ \|_I$ coincide on Re $Z(E)$. For, let $T \in$ Re $Z(E)$ and suppose that $\alpha = \|T\|_I = \inf \{\lambda \mid \lambda \in \mathbb{R}_+, \ T \leq \lambda I\}$. Then for each $\varepsilon \in (0, \alpha)$ there exists a $U \in$ Re $Z(E)$ such that $0 \leq U \leq I$, $U \neq 0$ and $TU \geq$ $\geq (\alpha - \varepsilon)U \geq 0$ i.e., $(\alpha - \varepsilon)U \ll TU$. Since $U \neq 0$, there exists an $x \in E$ such that $y = Ux \neq 0$. Then $(\alpha - \varepsilon)y \ll Ty$, so by Proposition 1.2 iii) $\|Ty\| \geq (\alpha - \varepsilon)\|y\|$. Consequently, $\|T\| \geq \alpha - \varepsilon$. The fact that $\alpha = \|T\|_I \leq \|T\|$ follows from Lemma 4.5 and AE5). ■

By Proposition 7.1 above, all operators A in Re $Z(E)$ are self-adjoint. In fact, $\|e^{itA}\| = 1$ for every $t \in \mathbb{R}$.

The operators in $Z(E)_+$ are diagonal in the sense that they leave invariant every hereditary cone of E. Consequently, for every $A \in Z(E)_+$, $x \ll y$ implies that $Ax \ll Ay$.

If T is an arbitrary operator in $Z(E)$ then $T(E_x) \subset E_x$ for every x so that every $v \in$ Ex E is an eigenvector for T. This connection between spectral theory and geometry will be used in section 10 to prove Hilbert-Schmidt type theorems for operators acting on Banach spaces. It is clear that when $Z(E) \setminus \{0\}$ contains no compact operator then Ex E must be the empty set.

7.3 PROPOSITION. *$Z(E)$ is a full subalgebra of $L(E,E)$* i.e., *if T^{-1} exists in $L(E,E)$, then $T^{-1} \in Z(E)$.*

Proof. Let $T \in Z(E)$ such that T^{-1} exists in $L(E,E)$. We shall prove that there exists a constant $\alpha > 0$ such that $|T| \geq \alpha I$. If this is not the case then for all $n \in \mathbb{N}^*$

we have $(I - n|T|)^+ > 0$ and thus there exist $u_n \in E$ and $\lambda_n > 0$ such that

$$v_n = (I - n|T|)^+ u_n \neq 0;\ v_n \ll \lambda_n u_n .$$

Since $T \in Z(E)$,

$$(I - n|T|)\, v_n = (I - n|T|)^+ v_n$$

which yields

$$v_n = n|T|\, v_n + (I - n|T|)^+ v_n .$$

Particularly, $n|T|\, v_n \ll v_n$ for all n. Then

$$\|v_n\| = \|T^{-1}Tv_n\| \leq \|T^{-1}\|\cdot\|Tv_n\| \leq 2\,\|T^{-1}\|\cdot\|\,|T|\, v_n\| \leq \|T^{-1}\|\cdot(2/n)\cdot\|v_n\|$$

which yields $\|T^{-1}\| \geq n/2$ for every $n \in \mathbb{N}^*$, a contradiction. We used the fact that $T^{\pm} \ll |T|$ and thus $\|Tv\| \leq 2\|\,|T|\, v\|$ for every $v \in E$. ■

7.4 PROPOSITION. *For each $x \in E$ the map $T \to Tx$ is a lattice morphism from $Z(E)_+$ into E.*

In fact, it suffices to prove that for every $S,T \in Z(E)^+$ we have $\sup\{S,T\}x = \sup\{Sx,Tx\}$ which follows the lemma below:

7.5 LEMMA. *Let $S,T \in$ Re $Z(E)$ and $x,y \in E$ such that $Sx \ll y$ and $Tx \ll y$. Then $\sup\{S,T\}x \ll y$.*

Proof. We shall identify S and T with their images in $C(\Omega, \mathbb{R})$. See Proposition 7.1 above. For $\varepsilon > 0$ given, consider the sets

$$\Omega_1 = \{\omega \mid \omega \in \Omega,\ S(\omega) \geq T(\omega) \text{ and } \Omega_2 = \{\omega \mid \omega \in \Omega,\ T(\omega) \geq S(\omega) + \varepsilon\}.$$

Then there exists a $U \in Z(E)$ such that $0 \leq U \leq I$, $U\omega = 1$ for $\omega \in \Omega_1$ and $U\omega = 0$ for $\omega \in \Omega_2$. Then

$$\sup\{S,T\} + \varepsilon I \geq US + (I - U)T \geq \sup\{S,T\} - \varepsilon I$$

i.e., $US + (I - U)T - \sup\{S,T\} + \varepsilon I \ll 2\varepsilon I$. Then

$$USx + (I - U)Tx - \sup\{S,T\}x + \varepsilon x \ll 2\varepsilon x$$

so by AE5) and the fact that $\varepsilon > 0$ is arbitrarily small we obtain that $\sup\{S,T\}x = USx + (I - U)Tx \ll y$. ■

7.6 PROPOSITION. *Suppose that there exists a weaker locally context Hausdorff topology τ on E such that:*

i) *every $\ll$ - downwards directed net of elements of E contains a τ - convergent subnet.*

ii) *every $\ll$ - interval $[u,v]$ is τ - closed.*

Then $Z(E)$ is order complete.

Proof. By Proposition 7.1 it suffices to consider downwards directed nets $(T_\alpha)_\alpha$ of elements of $Z(E)_+$. Then for each $x \in E$ the limit $Tx = \tau$ - $\lim T_\alpha x$ exists in E. It is clear that T is the $\ll$ - g.l.b. of $(T_\alpha)_\alpha$. ■

Proposition 7.6 applies in each of the following cases:

– the norm of E is $\ll$ - continuous

– $\ll = \ll_M$ and E is a dual Banach space.

See Corollary 6.3 above.

If E is a regularly ordered Banach space, then

$$Z_{o}(E) = \{T \mid T \in L(E,E),\ -\alpha I \leq T \leq \alpha I \text{ for a suitable } \alpha \geq 0\}.$$

This case was first considered by Wils [Wi] who noticed also the connection with the notion of center of a C^* - algebra.

The following special case is very illuminating. Let S be a compact Hausdorff space and $T \in Z_o(C(S, \mathbb{R}))_+$ i.e. $T \in L(C(S, \mathbb{R}), C(S, \mathbb{R}))$ and $0 \leq T \leq \alpha I$ for some $\alpha \geq 0$. We shall show that T is a multiplier i.e., T is of the form $Tx = \varphi \cdot x$ for some $0 \leq \varphi \leq \alpha$ in $C(S, \mathbb{R})$. In fact, for each $s \in S$, the functional $F_s : x \to (Tx)(s)$ satisfies the inequality

$$|F_s(x)| \leq \alpha\, |x(s)|,\ x \in C(S, \mathbb{R})$$

which yields a $\varphi(s) \in [0,\alpha]$ such that $F_s(x) = \varphi(s) \cdot x(s)$ for all $x \in C(S, \mathbb{R})$; φ belongs to $C(S, \mathbb{R})$ because $\varphi = T(1)$. Consequently the map

$$\Phi : Z_o(C(S, \mathbb{R})) \to C(S, \mathbb{R})$$

given by $\Phi(T) = T(1)$ is an algebraic lattice and isometric isomorphism. Notice also that $Z_o(C(S, \mathbb{R})) = Z_M(C(S, \mathbb{R}))$.

7.7 LEMMA. *Let E be a Banach lattice and $T \in Z_o(E)_+$. Then $|Tx| = T|x|$ for every $x \in E$ i.e., T is a lattice morphism.*

Proof. By hypotheses, there exists an $\alpha \geq 0$ such that $0 \leq T \leq \alpha I$. See also Lemma 1.0. Then $\alpha|x| = |Tx| + |\alpha x - Tx| \leq T|x| + (\alpha I - T)|x| = \alpha|x|$ which yields $|Tx| = T|x|$ for every $x \in E$. ■

An element $u > 0$ of a Banach lattice E is said to be **quasi-interior** provided that E_u is dense in E. If u is quasi-interior, then the map $T \to T \mid E_u$ is an algebraic lattice isomorphism from $Z(E)$ onto $Z(E_u)$. Since E_u is a space $C(S, \mathbb{R})$ we infer that $Z_o(E) \rightrightarrows E_u$ as Banach lattices. Consequently

$$Z_o(L^p(\mu, \mathbb{R})) \simeq L^\infty(\mu, \mathbb{R})$$

for every positive finite measure μ and every $p \in [1,\infty]$. Notice also that

$$Z_L(L^1(\mu, \mathbb{R})) = Z_o(L^1(\mu, \mathbb{R})).$$

Since $\ell^\infty(2, \mathbb{R})$ and $\ell^1(2, \mathbb{R})$ are isometric Banach spaces,

$$Z_L(\ell^\infty(2, \mathbb{R})) \simeq Z_L(\ell^1(2, \mathbb{R})) = Z_o(\ell^1(2, \mathbb{R})) \simeq \ell^\infty(2, \mathbb{R})$$

and $Z_M(\ell^\infty(2, \mathbb{R})) \simeq \ell^\infty(2, \mathbb{R})$.

8. CUNNINGHAM PROJECTIONS

The aim of this section is to discuss a class of idempotents canonically associated to a given AE-order relation « on a Banach space E.

8.1 *Definition.* A projection $P \in L(E,E)$ is said to be a (« -) **Cunningham projection** provided that Px « x for every $x \in E$.

The images of ($\ll$ -) Cunningham projections will be called ($\ll$ -) **summands**. Clearly, 0 and I are Cunningham projections. By AE1), if P is a Cunningham projection so is I-P.

We shall denote by $\mathbb{P}(E)$ the set of all ($\ll$ -) Cunningham projections on E; sub-scripts should be used in order to avoid any confusion.

8.2 PROPOSITION. *The Cunningham projections on E are precisely the idempotents of $Z(E)$.* ■

Proof. If $P \in \mathbb{P}(E)$, then $P \ll I$ and thus P is an idempotent of $Z(E)$. Conversely, let $P \in Z(E) = C(\Omega, \mathbb{K})$ such that $P^2 = P$. Then P is the characteristic function of an open-and-closed subset A of Ω. By Lemma 4.4 above, since $0 \leq \chi_A \leq 1$, it follows that $P = \chi_A \ll I$ in $Z(E)$.

The predecessors of our definition are the L and M - projections introduced by Cunningham. A projection $P \in L(E,E)$ is said to be an **L-projection** if $\|x\| = \|Px\| + \|x - Px\|$ and **M-projection** if $\|x\| = \|Px\| \vee \|x - Px\|$ for every $x \in E$. Clearly, the L-projections and the $\ll_L$-projections coincide. The corresponding result for $\ll_M$ is stated below:

8.3 PROPOSITION. *Let φ be an isometric vector norm with RDP on the Banach space E and let $P \in L(E,E)$ a projection. Then the following assertions are equivalent*:

i) *P is an $\ll_{M,\varphi}$ -projection*;

ii) $\varphi(Px + y) \leq \varphi(x + y) \vee \varphi(y)$ *for every* $x,y \in E$;

iii) $\varphi(x) = \varphi(Px) \vee \varphi(x - Px)$ *for every* $x \in E$.

Proof. Clearly, i) $\Leftrightarrow$ ii) and iii) $\Rightarrow$ ii).

ii) $\Rightarrow$ iii). Notice first that $\varphi(Px)$, $\varphi(x - Px) \leq \varphi(x)$ for every $x \in E$. On the other hand, $\varphi(Px + (I\text{-}P)y) \leq \varphi(y + P(x - y)) \leq \varphi(y) \vee \varphi(x)$ for every $x,y \in E$,, which yields $\varphi(x) = \varphi(P^2x + (I\text{-}P)^2x) \leq \varphi(Px) \vee \varphi(x - Px)$ for every $x \in E$. ■

Other examples of Cunningham projections are indicated below.

8.4 PROPOSITION. *Let E be a Banach lattice. Then the $\ll_o$ - Cunningham projections on E are precisely the band projections on E.*

Recall that a **band projection** on E is any positive projection P on E such that $|x - Px| \wedge |Py| = 0$ for every $x,y \in E$.

Proof. Clearly, the band projections are $\ll_o$ - Cunningham projections.

Let P be a $\ll_o$ - Cunningham projection. By Lemma 7.7, P and I - P are lattice morphisms so that in order to establish the relation $|x - Px| \wedge |Py| = 0$ it suffices to consider the case where $x \geq 0$ and $y \geq 0$. Or, in the later case,

$$0 \leq (x - Px) \wedge Py \leq Py \in \operatorname{Im} P$$
$$0 \leq (x - Px) \wedge Py \leq x - Px \in \operatorname{Ker} P$$

which yields $(x - Px) \wedge Py = 0$. ■

8.5 PROPOSITION. *Let H be a Hilbert space and $\mathcal{A}$ the von Neumann algebra generated by a self-adjoint operator $A \in L(H,H)$. Then the $\ll_{\mathcal{A}}$- Cunningham projections are precisely the orthogonal projections belonging to $\mathcal{A}$.*

Proof. Let $P \in \mathbb{P}_{\mathcal{A}}(H)$. Then for every $x \in H$ there exists a $B \in \mathcal{A}$ such that $0 \leq B \leq I$ and $Px = Bx$. Consequently $P = P^*$ and P belongs to the *wo* - closure of $\mathcal{A}$ i.e. to $\mathcal{A}$.

The converse is clear. ■

One can prove (see [*AE*]) that the *M*-summands of any von Neumann algebra $\mathcal{A}$ are the weak closed two-sided ideals of $\mathcal{A}$. The following two results describe geometric properties of Cunningham projections.

8.6 PROPOSITION. *Let P be a « - Cunningham projection on E. Then P maps every « - facial cone C of E into a « - facial cone P(C) contained C.*

By Lemma 4.1, either $P(C) = C$ or $P(C) \subset \mathrm{Fr}\, C$.

8.7 PROPOSITION. *Every two « - Cunningham projections commute.*

Proof. The image of every « - Cunningham projection is « - hereditary. In fact, $y \ll x \in \mathrm{Im}\, P$ yields $(I-P)y \ll (I-P)x = 0$, so that $y = Py \in \mathrm{Im}\, P$. Since the Cunningham projections leave invariant the hereditary cones, for each two Cunningham projections P and Q on E we have $PQ(E) \subset Q(E)$ i.e., $PQ = QPQ$. By replacing Q with $I - Q$ we obtain also that $P(I-Q) = (I-Q)P(I-Q)$ i.e., $QP = QPQ$. Consequently $PQ = QP$. ■

The following result shows that Cunningham projections are $\mathbb{R}$ - determined.

8.8 PROPOSITION. *Let E be a complex Banach space endowed with an AE-order relation «. Then E and $E_{\mathbb{R}}$, the underlying real space, have the same « - Cunningham projections and thus the same « - summands.*

Proof. It suffices to show that if $P \in L(E_{\mathbb{R}}, E_{\mathbb{R}})$ is a « - Cunningham projection then $P(\mathrm{i}Px) = \mathrm{i}\, P(x)$ for every $x \in E$.

For notice that the map $Q : x \to -\mathrm{i}P(\mathrm{i}x)$ defines a « - Cunningham projections commute, $P(\mathrm{i}P(\mathrm{i}x)) = \mathrm{i}P(\mathrm{i}P(x))$ for every $x \in E$.

Given $x \in E$, put $y_x = Px + \mathrm{i}P(\mathrm{i}P(x)) = -\mathrm{i}[\mathrm{i}P(x) - P(\mathrm{i}P(x))]$. Then $Py_x = y_x$ and $P(\mathrm{i}y_x) = 0$. Therefore $y_x = P(y_x + \mathrm{i}\, y_x) \ll (1+\mathrm{i})y_x$, so by Corollary 1.4 it follows that $(1-\mathrm{i})y_x \ll 2y_x$. Consequently $-\mathrm{i}y_x = (I-P)(y_x - \mathrm{i}y_x) \ll 2(I-P)y_x = 0$ i.e., $y_x = 0$. ■

8.9 PROPOSITION. *A closed subspace F of E is a « - summand if and only if there exists a closed subspace G of E (In fact $G = F^{\perp}$) such that $E = F \oplus G$ algebraically and $x \parallel y$ for every $x \in E$ and $y \in G$.*

Proof. In fact, if F is the image of the « - Cunningham projection P on E, then we can choose $G = (I-P)(E) = P(E)^{\perp}$. Conversely, the natural projection $P : F \oplus G \to F$ is linear and satisfies the condition $P \ll I$ i.e., P is a « - Cunningham projection . ■

In general, the complementary set $F^{\perp}$ of a subspace F may be not a subspace. If P is a Cunningham projection then $P(E)^{\perp} = (I-P)(E)$ i.e., the complementary set of a summand is also a summand.

8.10 COROLLARY. *Let F be a « - summand of E. Then there exists a unique « - projection P on E whose range is F.*

8.11 PROPOSITION. *The following assertions are equivalent for F a closed subspace of E:*

i) *F is a « - summand;*

ii) *F satisfies the following two conditions:*

ii$_1$) (Riesz decomposition property). *If $h \ll u + v$ in E and $h \in F$ then there exist j and k in F such that $h = j + k$, $j \ll u$, $k \ll v$;*

ii$_2$) *For each $x \in E$ the set $\{y \mid y \in F, y \ll x\}$ has a* l.u.b. *in F.*

iii) *F satisfies the conditions* ii$_1$) & ii$_2$) *above and also the interpolation property* (i.e., *if $m,n \ll x$ with $m,n \in F$ and $x \in E$ then there exists a $p \in F$ such that $m,n \ll p \ll x$*).

Proof. Clearly, i) $\Rightarrow$ iii) and iii) $\Rightarrow$ ii).

ii) $\Rightarrow$ i). For $x \in E$ put $Px = \sup \{y \mid y \in F, y \ll x\}$. Then $x - Px \in F^{\perp}$. In fact, if the contrary is true then it would exist a $z \in F$, $z \neq 0$ such that $z \ll x - Px$. Since $z \ll x - Px$, $Px \ll x$ and $x - Px \ll x$ it follows that $z + Px \ll Px$, a contradiction.

$F^{\perp}$ is a vector subspace. In fact, let $u, v \in F^{\perp}$ and $x \in F$ with $x \ll u + v$. Because of ii$_1$) and the fact that $F^{\perp}$ is hereditary we obtain that $x = 0$ and thus $u+v \in F^{\perp}$. Clearly $u \in F^{\perp}$ and $\alpha \in \mathbb{K}$ implies $\alpha u \in F^{\perp}$. Consequently $F^{\perp}$ is a vector subspace and the map $P : x \to Px$ is a $\ll$ - Cunningham projection. ■

For E an order complete Banach lattice and $\ll = \ll_o$, Proposition 8.11 reads as follows

8.12 PROPOSITION. (see [S2]). *Let E be an order complete Banach lattice. A closed subspace $\mathcal{J}$ of E is a projection band if and only if $\mathcal{J}$ is a closed lattice ideal such that $A \subset \mathcal{J}$ and* sup $A = x$ *exists in E implies $x \in \mathcal{J}$.*

9. BOOLEAN ALGEBRAS OF PROJECTIONS

Let E be a Banach space. By a **Boolean algebra of projections** on E we mean any Boolean algebra $\mathcal{B}$ of mutually commuting idempotents of $L(E,E)$ with respect to the following operations:

$$P \vee Q = P + Q - PQ$$
$$P \wedge Q = PQ$$
$$P^{\perp} = I - P.$$

A Boolean algebra $\mathcal{B}$ of projections on E is said to be **equicontinuous** provided that $\sup \{\|P\| \mid P \in \mathcal{B}\} < \infty$.

A Boolean algebra $\mathcal{B}$ of projections on E is said to be **Bade complete** provided that for every family $(P_\alpha)_\alpha$ of elements of $\mathcal{B}$ there exist $\vee P_\alpha$ and $\wedge P_\alpha$ in $\mathcal{B}$ and moreover

$$(\vee P_\alpha)(E) = \overline{\mathrm{Span}} \bigcup_\alpha P_\alpha(E)$$
$$(\wedge P_\alpha)(E) = \bigcap_\alpha P_\alpha(E) \ .$$

By Proposition 8.7 above, the set $\mathbb{P}(E)$ of all $\ll$ - Cunningham projections on E constitutes a Boolean algebra. These projections P are **bicontractive** in the sense that $\|P\| \leq 1$ and $\|2P - I\| \leq 1$.
See Propositions 7.1 and 8.1. Particular cases are

$\mathbb{P}_L(E)$, the Boolean algebra of all L - projections on E;

$\mathbb{P}_M(E)$, the Boolean algebra of all M - projections on E;

$\mathbb{P}_o(E)$, the Boolean algebra of all band projections on E (E is supposed to be a Banach lattice);

$\mathbb{P}_{\mathcal{A}}(H)$, the Boolean algebra of all orthogonal projections belonging to a commutative von Neumann algebra $\mathcal{A}$ of $L(H,H)$ (H being a Hilbert space).

$\mathbb{P}_L(E)$ and $\mathbb{P}_{\mathcal{A}}(H)$ are examples of Bade complete Boolean algebras of projections. $\mathbb{P}_o(E)$ is Bade complete provided that E is a Banach lattice with (o) - continuous norm.

$\mathbb{P}_M(E)$ is order complete provided that E is a dual Banach space.

As a matter of fact, all important examples of Boolean algebras of projections come through Alfsen-Effros theory via a renorming process described below.

9.1 LEMMA. *Let $\mathscr{B}$ be a Bade complete Boolean algebra of projections on E. Then $P_\alpha \uparrow P$ in $\mathscr{B}$ implies $P_\alpha \xrightarrow{so} P$.*

Proof. Let $x \in E$ and let $\varepsilon > 0$. Since $P(E) = \overline{\text{Span} \bigcup_\alpha P_\alpha(E)}$, there exists indices $\alpha_1,\ldots,\alpha_n$ and vectors $x_1,\ldots,x_n$ such that $P_{\alpha_i} x_i = x_i$ and $\left\| Px - \sum_{i=1}^{n} x_i \right\| < \varepsilon$.

Let $\alpha \geq \alpha_1,\ldots,\alpha_n$ and put $y = \sum_{i=1}^{n} x_i$. Then $P_\alpha(y) = y$ and

$$\begin{aligned} \| P_\alpha(x) - P(x) \| &\leq \| P_\alpha(x) - y \| + \| y - P(x) \| = \\ &= \| P_\alpha(P(x)-y) \| + \| y - P(x) \| < 2\varepsilon. \ \blacksquare \end{aligned}$$

9.2 LEMMA. (Bade [B1]; see also [DS], ch. XVII). *Let $\mathscr{B}$ be a Boolean algebra of projections such that for each sequence $(P_n)_n$ of $\mathscr{B}$ there exists $\vee_n P_n$ in $\mathscr{B}$. Then $\mathscr{B}$ is equicontinuous.*

Given an equicontinuous Boolean algebra $\mathscr{B}$ of projections on E we can consider the Banach algebra $\mathscr{C}(\mathscr{B})$ generated by it in $L(E,E)$. $\mathscr{C}(\mathscr{B})$ is a commutative Banach algebra with unit I, called the **Bade algebra** generated by $\mathscr{B}$.

By *P*roposition 8.2, $\mathscr{C}(\mathbb{P}(E)) \subset Z(E)$.

The other inclusion needs additional assumptions. In fact,

$$\mathscr{C}(\mathbb{P}_o(C([0,1],\mathbb{R}))) = \mathbb{R} \cdot I \text{ and } Z_o(C([0,1],\mathbb{R})) = C([0,1],\mathbb{R}).$$

9.3 LEMMA. *Suppose that $Z(E)$ is order complete. Then $\mathscr{C}(\mathbb{P}(E)) = Z(E)$.*

Proof. By Theorem 7.2, we can identify $Z(E)$ with a space $C(\Omega, \mathbb{K})$. Since $C(\Omega, \mathbb{K})$ is order complete, the subalgebra $\mathscr{A}$ (of all finite sums of the form $\sum \alpha_i \chi_{A_i}$ where A_i' s are mutually disjoint open-and-closed subsets of Ω) is norm dense; use Stone-Weierstrass approximation theorem. Or, $\mathscr{A}$ = Span $\mathbb{P}(E)$ and thus $\mathscr{C}(\mathbb{P}(E)) = Z(E)$. $\blacksquare$

It is worthwhile to mention that given an Boolean algebra $\mathscr{B}$ of equicontinuous projections on E, the map

$$x \to \|x\|_1 = \sup\{\|Px\|, \|2Px - x\| \mid P \in \mathscr{B}\}$$

is an equivalent norm on E and each $P \in \mathscr{B}$ is a bicontractive projection on $(E, \| \|_1)$.

Suppose now that $\mathscr{B}$ is a Boolean algebra of bicontractive projections on E and S is the Stone space of $\mathscr{B}$; $\mathscr{B}$ is isomorphic to the Boolean algebra $\mathscr{D}$ of all open-and-closed subsets of the compact Hausdorff space S. For $A \in \mathscr{D}$, we shall denote by P_A the corresponding projection in $\mathscr{B}$. Consider the linear span L, of all characteristic functions χ_A, for $A \in \mathscr{D}$. For each $f \in L, f \neq 0$, can be represented uniquely as a finite sum $f = \sum \alpha_i \chi_{A_i}$, where the sets $A_i \in \mathscr{D}$ are mutually disjoint and non empty. The mapping

$$\Phi : L \to \text{Span } \mathscr{B}$$

given by

$$\Phi\left(\sum \alpha_i \chi_{A_i}\right) = \sum \alpha_i P_{A_i}$$

belongs to $\mathcal{A}$ and thus it is a $\ll_{\mathcal{A}}$ - Cunningham projection. Since (HS) yields

$$A = \sum_{\{\lambda_n \in \sigma_p(A)\}} \lambda_n P_n$$

is the norm topology of $L(H,H)$, it follows that $A \in \operatorname{Re} Z_{\mathcal{A}}(H)$.

It is remarkable that the result above remains valid in the general setting.

10.1 HILBERT-SCHIMDT GENERALIZED THEOREM. i) *Let E be a Banach space endowed with an AE-order relation «, such that the norm of E is « - continuous. Then for every compact operator* $A \in \operatorname{Re} Z(E)$ *there exist a real sequence* $(\alpha_n)_n \in c_0$ *and a sequence* $(\alpha_n)_n$ *of mutually disjoint finite rank Cunningham projections on E such that* $A(x) = \sum \alpha_n P_n(x)$ *for every* $x \in E$.

ii) *Conversely, every operator* $A \in L(E,E)$ *that admits such a representation is compact and belongs to* Re $Z(E)$.

The assertion ii) is an easy consequence of axiom AE4). In fact, it suffices to consider representations

$$A(x) = \sum \alpha_n P_n(x)$$

where $(\alpha_n)_n \in (c_0)_+$ and $(P_n)_n$ are sequences of mutually disjoint finite rank Cunningham projections. Then by AE4),

$$\left\| A(x) - \sum_{k=0}^{n} \alpha_k P_k(x) \right\| = \left\| \sum_{k=n+1}^{\infty} \alpha_k P_k(x) \right\| \leq$$
$$\leq \left(\sup_{k \geq n+1} |\alpha_k| \right) \cdot \|x\|$$

for every $n \in \mathbb{N}$ and $x \in E$.

The assertion i) will be obtained by cutting A in smaller pieces. We shall need the following special case of Glicksberg - deLeeuw decomposition theorem (see [Kr], 2.4.4, for the general case):

10.2 LEMMA. *Suppose that E is endowed with an AE-order relation «, that makes the norm of E « - continuous. Let* $A \in \operatorname{Re} Z(E)$, *with* $\|A\| \leq 1$. *Then* $(A^n)_n$ *converges strongly to a Cunningham projection P such that* $PA = AP = P$ *and* $\operatorname{Im} P = \operatorname{Ker}(I - A)$.

Proof. By Theorem 7.2 ii) above, we have $A \ll I$. Since the norm of E is « - continuous, the limit $Px = \lim_{n\to\infty} A^n x$ exists for each $x \in E$. That gives raise to an operator P such that $P \ll I$. We shall show that P is also a projection (and thus it is a Cunningham projection). For $x \in E$ and $\varepsilon > 0$ given, choose an $n \in \mathbb{N}$ for which

$$\|A^k x - Px\| < \varepsilon/3 \text{ and } \|A^m Px - P^2 x\| < \varepsilon/3$$

if $k, m \geq n$. Then

$$\|P^2 x - Px\| \leq \|P^2 x - A^n Px\| + \|A^n Px - A^n A^n x\| +$$
$$+ \|A^n A^n Px - Px\| <$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

which yields the fact that $P^2 = P$. The others assertions in the statement of Lemma 10.2 are straightforward.

10.3 COROLLARY. *Under the above assumptions on E, for each compact nonnull operator A in* Re $Z(E)$ *there exists a finite rank nonnull Cunningham projection P such that* $PA = AP = P$ *and* $\operatorname{Im} P = \operatorname{Ker}(\|A\| \cdot I - A)$.

Proof. By Theorem 7.2 ii) above, we have $A/\|A\| \ll I$, so the existence of P is assured by Lemma 10.2. Since A is compact and self-adjoint, $\|A\|$ is an eigenvalue and thus $\operatorname{Im} P \neq \{0\}$. P is finite rank, as being compact. ■

Now, to derive assertion i) in Theorem 10.1 from Corollary 10.3, we have to choose $\alpha_0 = \|A\|$ and $P_0 = P$. Because $A = AP + A(I - P) = \alpha_0 P_0 + A(I - P_0)$ the cutting procedure will continue by replacing A by $A(I - P_0)$.

An alternative proof of Theorem 10.1 can be found in [N5].

By combining Theorems 5.10 and 10.1 we obtain the following:

10.4 PROPOSITION. *Under the assumptions of Theorem 10.1,*

$$\overline{\operatorname{Im} A} = \overline{\operatorname{Span}}\left[(\operatorname{Im} A) \cap (Ex\, E)\right].$$

In other words, $\overline{\operatorname{Im} A}$ is generated by its « - extreme points.

10.5 COROLLARY. *If E has « - continuous norm and* $Z_{\ll}(E)$ *contains a compact nonnull operator, then* $\operatorname{Ex}_{\ll} E$ *is nonempty.*

Since $L^1[0,1]$ has no atom, from Corollary 10.5 we infer that the only compact operator $T : L^1[0,1] \to L^1[0,1]$ with $0 \leq T \leq I$ is $T = 0$.

The converse of Corollary 10.5 is false. Think at the case where $E = L^2[0,1]$ and $\ll \, = \, \ll_L$.

Suppose now E is a Banach lattice whose order intervals $[0,x]$ are compact. Then a result due to Walsh [Wh] asserts that E is discrete (i.e., it has an unconditional basis consisting of atoms). To derive this fact from Theorem 10.1 it suffices to restrict ourseves to the separable case and to remark the existence of a positive element u in E and of a compact operator A in $L(E,E)$ such that

$$E = \overline{\operatorname{Im} A} = \overline{\operatorname{Span}}[0,u] \text{ and } 0 \leq A \leq I.$$

A natural question arising in this setting is the following:

10.6 **Problem.** Let E be a Banach space endowed with an AE-order relation « and let $z \in E$ such that the « - order interval $[0, z]$ is compact. Can each $x \in C(z)$ be represented as an unconditional convergent series

$$x = \sum c_n e_n$$

with $c_n \geq 0$ and $e_n \in \operatorname{Ex} E \cap C(z)$ suitably chosen?

The classical Hilbert-Schmidt theorem covers all orthonormal expansions because every orthonormal basis of a Hilbert space arises as the fundamental system of eigenvectors of a suitable self - adjoint compact operator $A \in L(H,H)$. What type of decomposition Theorem 10.1 yields?

Under the assumptions of Theorem 10.1, $E = \overline{\operatorname{Im} A} \oplus \operatorname{Ker} A$ constitutes an unconditional decomposition of lattice constant 1 and the subsequent decomposition of $\overline{\operatorname{Im} A}$ A constitutes an unconditionally finite dimensional decomposition of lattice constant 1. That is explained in [N5]. One can prove that (via a renorming process) all complemented subspaces having an unconditional finite dimensional decomposition arise in this way i.e., via Alfsen - Effros theory.

Our final remark is that one can rephrase Theorem 10.1 in terms independent of any order-theoretical structure. In fact, by Theorem 9.5, it can be restated as follows:

10.7 THEOREM. *Let $\mathcal{B}$ be a Bade complete Boolean algebra of projections on the Banach space E and let $A \in \mathscr{C}(\mathcal{B})$ a compact operator. Then there exist a sequence $(\alpha_n)_n \in c_o$ and a sequence $(P_n)_n$ of finite rank mutually disjoint projections in such that*

$$A = \sum \alpha_n P_n \,.$$

in the norm topology of L(E,E).

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